JOINT SIMILARITY TO OPERATORS IN NONCOMMUTATIVE VARIETIES

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ABSTRACT. In this paper we solve several problems concerning joint similarity to n-tuples of operators in noncommutative varieties $\mathcal{V}^m_{f,\mathcal{P}}(\mathcal{H}) \subset B(\mathcal{H})^n$, $m \geq 1$, associated with positive regular free holomorphic functions f in n noncommuting variables and with sets $\mathcal{P}$ of noncommutative polynomials in n indeterminates, where $B(\mathcal{H})$ is the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. In particular, if $f = X_1 + \cdots + X_n$ and $\mathcal{P} = \{0\}$, the elements of the corresponding variety can be seen as noncommutative multivariable analogues of Agler's m-hypercontractions.

We introduce a class of generalized noncommutative Berezin transforms and use them to solve operator inequalities associated with noncommutative varieties $\mathcal{V}^m_{f,\mathcal{P}}(\mathcal{H})$. We point out a very strong connection between the cone of their positive solutions and the joint similarity problems. Several classical results concerning the similarity to contractions have analogues in our noncommutative multivariable setting. When $\mathcal{P}$ consists of the commutators $X_iX_j - X_jX_i$, $i, j \in \{1, \ldots, n\}$, we obtain commutative versions of these results. We remark that, in the particular case when $n = m = 1$, $f = X$, and $\mathcal{P} = \{0\}$, we recover the corresponding similarity results obtained by Sz.-Nagy, Rota, Foiaş, de Branges-Rovnyak, and Douglas.

We use some of the results of this paper to provide Wold type decompositions and triangulations for n-tuples of operators in noncommutative varieties $\mathcal{V}^1_{f,\mathcal{P}}(\mathcal{H})$, which parallel the classical Sz.-Nagy–Foiaş triangulations for contractions but also provide new proofs. As consequences, we prove the existence of joint invariant subspaces for certain classes of operators in $\mathcal{V}^1_{f,\mathcal{P}}(\mathcal{H})$.

INTRODUCTION

Let $B(\mathcal{H})$ denote the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. Two operators $A, B \in B(\mathcal{H})$ are called similar if there is an invertible operator $S \in B(\mathcal{H})$ such that $A = S^{-1}BS$. The problem of characterizing the operators similar to contractions, i.e., the operators in the unit ball

$$[B(\mathcal{H})]_1 := \{X \in B(\mathcal{H}) : \ XX^* \leq I\},$$

or similar to special contractions such as parts of shifts, isometries, unitaries, etc., has been considered by many authors and has generated deep results in operator theory and operator algebras. We shall mention some of the classical results on similarity that strongly influenced us in writing this paper.

In 1947, Sz.-Nagy [30] found necessary and sufficient conditions for an operator to be similar to a unitary operator. In particular, an operator T is similar to an isometry if and only if there are constants $a, b > 0$ such that

$$a\|h\| \leq \|T^n h\| \leq b\|h\|, \qquad h \in \mathcal{H}, n \in \mathbb{N}.$$

The fact that the unilateral shift on the Hardy space $H^2(\mathbb{T})$ plays the role of *universal model* in $B(\mathcal{H})$ was discovered by Rota [29]. Rota's model theorem asserts that any operator with spectral radius less than one is similar to a contraction, or more precisely, to a part of a backward unilateral shift. This result was refined furthermore by Foiaş [9] and by de Branges and Rovnyak [6], who proved that every strongly stable contraction is unitarily equivalent to a part of a backward unilateral shift.

It is well-known that if $T \in B(\mathcal{H})$ is similar to a contraction then, due to the von Neumann inequality [33], it is polynomially bounded, i.e., there is a constant $C > 0$ such that, for any polynomial p,

$$\|p(T)\| \leq C\|p\|_\infty,$$

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where $\|p\|_\infty := \sup_{|z|=1} |p(z)|$. A remarkable result obtained by Paulsen [14] shows that similarity to a contraction is equivalent to complete polynomial boundedness. Halmos' famous similarity problem [11] asked whether any polynomially bounded operator is similar to a contraction. This long standing problem was answered by Pisier [16] in a remarkable paper where he shows that there are polynomially bounded operators which are not similar to contractions. For more information on similarity problems and completely bounded maps we refer the reader to the excellent books by Pisier [17] and Paulsen [15].

In the noncommutative multivariable setting, joint similarity problems to row contractions, i.e., n-tuples of operators in the unit ball

$$[B(\mathcal{H})^n]_1 := \{(X_1, \dots, X_n) \in B(\mathcal{H})^n : \ X_1X_1^* + \cdots + X_nX_n^* \leq I\},$$

were considered by Bunce [3], the author (see [18], [22], [23], [26]), and recently by Douglas, Foiaş, and Sarkar [8]. In this setting, the *universal model* for the unit ball $[B(\mathcal{H})^n]_1$ is the n-tuple $(S_1, \dots, S_n)$ of left creation operators on the full Fock space with n generators.

To put our work in perspective we need some notation. Let $\mathbb{F}_n^+$ be the unital free semigroup on n generators $g_1, \dots, g_n$ and the identity g_0. The length of $\alpha \in \mathbb{F}_n^+$ is defined by $|\alpha| := 0$ if $\alpha = g_0$ and $|\alpha| := k$ if $\alpha = g_{i_1} \cdots g_{i_k}$, where $i_1, \dots, i_k \in \{1, \dots, n\}$. If $X := (X_1, \dots, X_n) \in B(\mathcal{H})^n$ we denote $X_\alpha := X_{i_1} \cdots X_{i_k}$ and $X_{g_0} := I_{\mathcal{H}}$, the identity on $\mathcal{H}$.

In [28] (case $m = 1$) and [25] (case $m \geq 2$), we studied more general noncommutative domains

$$\mathbf{D}_p^m(\mathcal{H}) := \{X := (X_1, \dots, X_n) \in B(\mathcal{H})^n : \ (id - \Phi_{p,X})^s(I) \geq 0 \ \text{ for } \ s = 1, \dots, m\},$$

where id is the identity map on $B(\mathcal{H})$,

$$\Phi_{p,X}(Y) := \sum_{|\alpha| \geq 1} a_\alpha X_\alpha Y X_\alpha^*, \qquad Y \in B(\mathcal{H}),$$

and $p = \sum_{|\alpha| \geq 1} a_\alpha X_\alpha$ is a positive regular noncommutative polynomial, i.e., its coefficients are positive scalars and $a_\alpha > 0$ if $\alpha \in \mathbb{F}_n^+$ with $|\alpha| = 1$. We remark that if $q = X_1 + \cdots + X_n$ and $m \geq 1$, then $\mathbf{D}_q^m(\mathcal{H})$ is a starlike domain which concides with the set of all row contractions $(X_1, \dots, X_n) \in [B(\mathcal{H})^n]_1$ satisfying the positivity condition

$$\sum_{k=0}^m (-1)^k \binom{m}{k} \sum_{|\alpha|=k} X_\alpha X_\alpha^* \geq 0.$$

The elements of the domain $\mathbf{D}_q^m(\mathcal{H})$ can be seen as multivariable noncommutative analogues of Agler's m-hypercontractions [1]. The case $n = 1$ was recently studied by Olofsson ([12], [13]). We showed ([28], [25]) that each domain $\mathbf{D}_p^m(\mathcal{H})$ has a *universal model* $(W_1, \dots, W_n)$ of *weighted left creation operators* acting on the full Fock space with n generators. The study of the domain $\mathbf{D}_p^m(\mathcal{H})$ and the dilation theory associated with it are close related to the study of the weighted shifts $W_1, \dots, W_n$, their joint invariant subspaces, and the representations of the algebras they generate: the domain algebra $\mathcal{A}_n(\mathbf{D}_p^m)$, the Hardy algebra $F_n^\infty(\mathbf{D}_p^m)$, and the C^*-algebra $C^*(W_1, \dots, W_n)$.

In the present paper, we consider problems of joint similarity to classes of n-tuples of operators in noncommutative domains $\mathbf{D}_p^m(\mathcal{H})$, $m \geq 1$, and noncommutative varieties

$$\mathcal{V}_{p,\mathcal{P}}^m(\mathcal{H}) := \left\{(X_1, \dots, X_n) \in \mathbf{D}_p^m(\mathcal{H}) : \ q(X_1, \dots, X_n) = 0 \quad \text{for any} \quad q \in \mathcal{P}\right\},$$

where $\mathcal{P}$ is a family of noncommutative polynomials in n indeterminates.

In Section 1, expanding on the author's work ([25], [27], [28]) on noncommutative Berezin transforms, we introduce a new class of generalized Berezin transforms which will play an important role in this paper. Given $A := (A_1, \dots, A_n) \in B(\mathcal{H})^n$, our similarity problems to n-tuples of operators in the noncommutative variety $\mathcal{V}_{p,\mathcal{P}}^m(\mathcal{H})$ are linked to the noncommutative cone $C(p, A)^+$ of all positive operators $D \in B(\mathcal{H})$ such that

$$(id - \Phi_{p,A})^s(D) \geq 0, \qquad s = 1, \dots, m.$$

For example, $(A_1, \dots, A_n)$ is jointly similar to an n-tuple of operators in $\mathcal{V}_{p,\mathcal{P}}^m(\mathcal{H})$ if and only if there is an invertible operator in $C(p, A)^+$. Under natural conditions, we show that there is a one-to-one correspondence between the elements of the noncommutative cone $C(p, A)^+$ and a class of generalized Berezin transforms, to be introduced.

In Section 2, a pure version of the above-mentioned result is established, even in a more general setting. In particular, when $m = 1$ and $T := (T_1, \ldots, T_n) \in \mathcal{V}^1_{p,\mathcal{P}}(\mathcal{H})$ is *pure*, i.e., $\Phi^k_{p,T}(I) \to 0$ strongly, as $k \to \infty$, we determine the noncommutative cone $C(p, T)^+$ by showing that all its elements have the form $P_\mathcal{H}\Psi\Psi^*|_\mathcal{H}$, where Ψ is a multi-analytic operator with respect to the universal n-tuple $(B_1, \ldots, B_n)$ associated with the variety $\mathcal{V}^1_{p,\mathcal{P}}(\mathcal{H})$. More precisely, $\Psi \in R^\infty_n(\mathcal{V}^1_{p,\mathcal{P}})\bar{\otimes}B(\mathcal{K}, \mathcal{K}')$ for some Hilbert spaces $\mathcal{K}$ and $\mathcal{K}'$, where $R^\infty_n(\mathcal{V}^1_{p,\mathcal{P}})$ is the commutant of the noncommutative Hardy algebra $F^\infty_n(\mathcal{V}^1_{p,\mathcal{P}})$. We remark that in the particular case when $n = m = 1$, $p = X$, $\mathcal{P} = \{0\}$, and $\Phi_{p,T}(X) := TXT^*$ with $\|T\| \leq 1$, the corresponding cone $C(p, T)^+$ was studied by Douglas in [7] and by Sz.-Nagy and Foiaş [32] in connection with T-Toeplitz operators (see also [4] and [5]).

In Section 3, we provide necessary and sufficient conditions for an n-tuple $A := (A_1, \ldots, A_n) \in B(\mathcal{H})^n$ to be jointly similar to an n-tuple of operators $T := (T_1, \ldots, T_n)$ in the noncommutative variety $\mathcal{V}^m_{p,\mathcal{P}}(\mathcal{H})$ or the distinguished sets

$$\left\{X \in \mathcal{V}^m_{p,\mathcal{P}}(\mathcal{H}) : \ (id - \Phi_{p,X})^m(I) = 0\right\} \quad \text{and} \quad \left\{X \in \mathcal{V}^m_{p,\mathcal{P}}(\mathcal{H}) : \ (id - \Phi_{p,X})^m(I) > 0\right\},$$

where $\mathcal{P}$ is a set of noncommutative polynomials. To give the reader a flavor of our results, we shall be a little bit more precise. Given $(A_1, \ldots, A_n) \in B(\mathcal{H})^n$, we find necessary and sufficient conditions for the existence of an invertible operator $Y : \mathcal{H} \to \mathcal{G}$ such that

$$A^*_i = Y^{-1}[(B^*_i \otimes I_\mathcal{H})|_\mathcal{G}]Y, \qquad i = 1, \ldots, n$$

where $\mathcal{G} \subseteq \mathcal{N}_\mathcal{P} \otimes \mathcal{H}$ is an invariant subspace under each operator $B^*_i \otimes I_\mathcal{H}$ and $(B_1, \ldots, B_n)$ is the universal model associated with the noncommutative variety $\mathcal{V}^m_{f,\mathcal{P}}(\mathcal{H})$. In particular, we obtain an analogue of Foiaş [9] and de Branges–Rovnyak [6] model theorem, for pure n-tuples of operators in $\mathcal{V}^m_{f,\mathcal{P}}(\mathcal{H})$. We also obtain the following Rota type [29] model theorem for the noncommutative variety $\mathcal{V}^m_{f,\mathcal{P}}(\mathcal{H})$. If $A := (A_1, \ldots, A_n) \in B(\mathcal{H})^n$ is such that $q(A_1, \ldots, A_n) = 0$ for $q \in \mathcal{P}$ and

$$\sum_{k=0}^{\infty} \binom{k+m-1}{m-1} \Phi^k_{p,A}(I) \leq bI$$

for some constant $b > 0$, then the above-mentioned joint similarity holds. Moreover, we prove that the joint spectral radius $r_p(A_1, \ldots, A_n) < 1$ if and only if $(A_1, \ldots, A_n)$ is jointly similar to an n-tuple $T := (T_1, \ldots, T_n) \in \mathcal{V}^m_{p,\mathcal{P}}(\mathcal{H})$ with $(id - \Phi_{p,T})^m(I) > 0$, i.e., positive invertible operator.

We also provide necessary and sufficient conditions for an n-tuple $A := (A_1, \ldots, A_n) \in B(\mathcal{H})^n$ to be jointly similar to an n-tuple of operators $T := (T_1, \ldots, T_n) \in \mathcal{V}^m_{p,\mathcal{P}}(\mathcal{H})$ with $(id - \Phi_{p,T})^m(I) = 0$. Our noncommutative analugue of Sz.-Nagy's similarity result [30] asserts that there is an invertible operator $Y \in B(\mathcal{H})$ such that $A_i = Y^{-1}T_iY$, $i = 1, \ldots, n$, if and only if there exist positive constants $0 < c \leq d$ such that

$$cI \leq \Phi^k_{p,A}(I) \leq dI, \qquad k \in \mathbb{N}.$$

In particular, we obtain a multivariable analogue of Douglas' similarity result [7].

If $(A_1, \ldots, A_n) \in B(\mathcal{H})^n$ is jointly similar to an n-tuple of operators in a *radial noncommutative variety* $\mathcal{V}^m_{p,\mathcal{P}}(\mathcal{H})$, where $\mathcal{P}$ is a set of homogeneous noncommutative polynomials, then the polynomial calculus $g(B_1, \ldots, B_n) \mapsto g(A_1, \ldots, A_n)$ can be extended to a completely bounded map on the noncommutative variety algebra $\mathcal{A}_n(\mathcal{V}^m_{p,\mathcal{P}})$, the norm closed algebra generated by $B_1, \ldots, B_n$ and the identity. Using Paulsen's similarity result [14], we can prove that the converse is true if $m = 1$, but remains an open problem if $m \geq 2$.

In Section 4, we obtain Wold type decompositions and prove the existence of triangulations of type

$$\begin{pmatrix} C_{\cdot 0} & 0 \\ * & C_{\cdot 1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} C_c & 0 \\ * & C_{cnc} \end{pmatrix}$$

for any n-tuple of operators in the noncommutative variety $\mathcal{V}^1_{p,\mathcal{P}}(\mathcal{H})$, which parallel the Sz.-Nagy–Foiaş [31] triangulations for contractions. The proofs seem to be new even in the classical case $n = 1$, since they don't involve, at least explicitly, the dilation space for contractions. As consequences, we prove the existence of joint invariant subspaces for certain classes of operators in $\mathcal{V}^1_{p,\mathcal{P}}(\mathcal{H})$.

We should mention that the results of this paper are presented in a more general setting when the polynomials p in the definition of $\mathcal{V}^m_{p,\mathcal{P}}(\mathcal{H})$ is replaced by positive regular free holomorphic functions.

1. Generalized noncommutative Berezin transforms and the cone $C(f,A)^+$

In this section, we introduce a class of generalized Berezin transforms which will play an important role in this paper. We use them to study the noncommutative cone $C(f,A)^+$ of all positive solutions of the operator inequalities

$$(id - \Phi_{f,A})^s(X) \geq 0, \qquad s = 1, \ldots, m.$$

First, we recall ([25], [28]) the construction of the universal model associated with the noncommutative domain $\mathbf{D}_f^m(\mathcal{H})$, $m \geq 1$. Throughout this paper, we assume that $f := \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha X_\alpha$, $a_\alpha \in \mathbb{C}$, is a *positive regular free holomorphic function* in n variables $X_1, \ldots, X_n$. This means

(i) $\limsup_{k\to\infty} \left(\sum_{|\alpha|=k} |a_\alpha|^2\right)^{1/2k} < \infty$,
(ii) $a_\alpha \geq 0$ for any $\alpha \in \mathbb{F}_n^+$, $a_{g_0} = 0$, and $a_{g_i} > 0$ for $i = 1, \ldots, n$.

Given $m \in \mathbb{N} := \{1, 2, \ldots\}$ and a positive regular free holomorphic function f as above, we define the noncommutative domain $\mathbf{D}_f^m$ whose representation on a Hilbert space $\mathcal{H}$ is

$$\mathbf{D}_f^m(\mathcal{H}) := \{X := (X_1, \ldots, X_n) \in B(\mathcal{H})^n : \ (id - \Phi_{f,X})^s(I) \geq 0 \ \text{ for } \ s = 1, \ldots, m\},$$

where $\Phi_{f,X} : B(\mathcal{H}) \to B(\mathcal{H})$ is given by

$$\Phi_{f,X}(Y) := \sum_{k=1}^{\infty} \sum_{|\alpha|=k} a_\alpha X_\alpha Y X_\alpha^*, \qquad Y \in B(\mathcal{H}),$$

and the convergence is in the weak operator topology. $\mathbf{D}_f^m(\mathcal{H})$ can be seen as a noncommutative Reinhardt domain, i.e., $(e^{i\theta_1}X_1, \ldots, e^{i\theta_n}X_n) \in \mathbf{D}_f^m(\mathcal{H})$ for any $(X_1, \ldots, X_n) \in \mathbf{D}_f^m(\mathcal{H})$ and $\theta_1, \ldots, \theta_n \in \mathbb{R}$.

Let H_n be an n-dimensional complex Hilbert space with orthonormal basis $e_1, \ldots, e_n$, where $n \in \mathbb{N}$ or $n = \infty$. We consider the full Fock space of H_n defined by

$$F^2(H_n) := \bigoplus_{k \geq 0} H_n^{\otimes k},$$

where $H_n^{\otimes 0} := \mathbb{C}1$ and $H_n^{\otimes k}$ is the (Hilbert) tensor product of k copies of H_n. Set $e_\alpha := e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$ if $\alpha = g_{i_1} g_{i_2} \cdots g_{i_k} \in \mathbb{F}_n^+$ and $e_{g_0} := 1$. It is clear that $\{e_\alpha : \alpha \in \mathbb{F}_n^+\}$ is an orthonormal basis of $F^2(H_n)$. Define the left creation operators $S_i : F^2(H_n) \to F^2(H_n)$, $i = 1, \ldots, n$, by $S_i f := e_i \otimes f$, $f \in F^2(H_n)$.

Let $D_i : F^2(H_n) \to F^2(H_n)$, $i = 1, \ldots, n$, be the diagonal operators given by

$$D_i e_\alpha := \sqrt{\frac{b_\alpha^{(m)}}{b_{g_i\alpha}^{(m)}}}\, e_\alpha, \qquad \alpha \in \mathbb{F}_n^+,$$

where

$$b_{g_0}^{(m)} := 1 \quad \text{ and } \quad b_\alpha^{(m)} := \sum_{j=1}^{|\alpha|} \sum_{\substack{\gamma_1 \cdots \gamma_j = \alpha \\ |\gamma_1| \geq 1, \ldots, |\gamma_j| \geq 1}} a_{\gamma_1} \cdots a_{\gamma_j} \binom{j+m-1}{m-1} \qquad \text{if } \ \alpha \in \mathbb{F}_n^+, \ |\alpha| \geq 1. \tag{1.1}$$

We have

$$\|D_i\| = \sup_{\alpha \in \mathbb{F}_n^+} \sqrt{\frac{b_\alpha^{(m)}}{b_{g_i\alpha}^{(m)}}} \leq \frac{1}{\sqrt{a_{g_i}}}, \qquad i = 1, \ldots, n.$$

Define the *weighted left creation operators* $W_i : F^2(H_n) \to F^2(H_n)$, $i = 1, \ldots, n$, associated with the noncommutative domain $\mathbf{D}_f^m$ by setting $W_i := S_i D_i$, where $S_1, \ldots, S_n$ are the left creation operators on the full Fock space $F^2(H_n)$. Note that

$$W_i e_\alpha = \frac{\sqrt{b_\alpha^{(m)}}}{\sqrt{b_{g_i\alpha}^{(m)}}}\, e_{g_i\alpha}, \qquad \alpha \in \mathbb{F}_n^+.$$

One can easily see that

$$W_\beta e_\gamma = \frac{\sqrt{b_\gamma^{(m)}}}{\sqrt{b_{\beta\gamma}^{(m)}}} e_{\beta\gamma} \quad \text{and} \quad W_\beta^* e_\alpha = \begin{cases} \frac{\sqrt{b_\gamma^{(m)}}}{\sqrt{b_\alpha^{(m)}}} e_\gamma & \text{if } \alpha = \beta\gamma \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

for any $\alpha, \beta \in \mathbb{F}_n^+$. According to Theorem 1.3 from [25], the weighted left creation operators $W_1, \ldots, W_n$ associated with $\mathbf{D}_f^m$ have the following properties:

(i) $\sum_{k=1}^\infty \sum_{|\beta|=k} a_\beta W_\beta W_\beta^* \leq I$, where the convergence is in the strong operator topology;
(ii) $(id - \Phi_{f,W})^m (I) = P_\mathbb{C}$, where $P_\mathbb{C}$ is the orthogonal projection from $F^2(H_n)$ onto $\mathbb{C}1 \subset F^2(H_n)$, and $\lim_{p\to\infty} \Phi_{f,W}^p(I) = 0$ in the strong operator topology.

The n-tuple $(W_1, \ldots, W_n) \in \mathbf{D}_f^m(F^2(H_n))$ plays the role of universal model for the noncommutative domain $\mathbf{D}_f^m$. The domain algebra $\mathcal{A}_n(\mathbf{D}_f^m)$ associated with the noncommutative domain $\mathbf{D}_f^m$ is the norm closure of all polynomials in $W_1, \ldots, W_n$, and the identity, while the Hardy algebra $F_n^\infty(\mathbf{D}_f^m)$ is the SOT-(WOT-, or w^*-) version.

We remark that, one can also define the *weighted right creation operators* $\Lambda_i : F^2(H_n) \to F^2(H_n)$ by setting $\Lambda_i := R_i G_i$, $i = 1, \ldots, n$, where $R_1, \ldots, R_n$ are the right creation operators on the full Fock space $F^2(H_n)$ and each diagonal operator G_i is defined by

$$G_i e_\alpha := \sqrt{\frac{b_\alpha^{(m)}}{b_{\alpha g_i}^{(m)}}} e_\alpha, \quad \alpha \in \mathbb{F}_n^+,$$

where the coefficients $b_\alpha^{(m)}$, $\alpha \in \mathbb{F}_n^+$, are given by relation (1.1). It turns out that $(\Lambda_1, \ldots, \Lambda_n)$ is in the noncommutative domain $\mathbf{D}_{\tilde{f}}^m(F^2(H_n))$, where $\tilde{f} := \sum_{|\alpha|\geq 1} a_{\tilde{\alpha}} X_\alpha$ and $\tilde{\alpha} = g_{i_k} \cdots g_{i_1}$ denotes the reverse of $\alpha = g_{i_1} \cdots g_{i_k} \in \mathbb{F}_n^+$. Moreover, $W_i \Lambda_j = \Lambda_j W_i$ and $U^* \Lambda_i U = W_i$, $i = 1, \ldots, n$, where $U \in B(F^2(H_n))$ is the unitary operator defined by equation $U e_\alpha := e_{\tilde{\alpha}}$, $\alpha \in \mathbb{F}_n^+$. Consequently, we have

$$F_n^\infty(\mathbf{D}_f^m)' = R_n^\infty(\mathbf{D}_f^m) \ \text{ and } \ R_n^\infty(\mathbf{D}_f^m)' = F_n^\infty(\mathbf{D}_f^m),$$

where $'$ stands for the commutant and $R_n^\infty(\mathbf{D}_f^m)$ is the SOT-(WOT-, or w^*-) closure of all polynomials in $\Lambda_1, \ldots, \Lambda_n$, and the identity. More on these noncommutative Hardy algebras can be found in [19], [25], and [28].

In what follows, we introduce a noncommutative Berezin kernel associated with any quadruple (f, m, A, R) satisfying the following compatibility conditions:

(i) $f := \sum_{|\alpha|\geq 1} a_\alpha X_\alpha$ is a positive regular free holomorphic function and $m \in \mathbb{N}$;
(ii) $A := (A_1, \ldots, A_n) \in B(\mathcal{H})^n$ is such that $\sum_{|\alpha|\geq 1} a_\alpha A_\alpha A_\alpha^*$ is SOT-convergent;
(iii) $R \in B(\mathcal{H})$ is a positive operator such that

$$\sum_{k=0}^\infty \binom{k+m-1}{m-1} \Phi_{f,A}^k(R) \leq bI,$$

for some constant $b > 0$.

The noncommutative Berezin kernel associated with the compatible quadruple (f, m, A, R) is the operator $K_{f,A,R}^{(m)} : \mathcal{H} \to F^2(H_n) \otimes \overline{R^{1/2}(\mathcal{H})}$ given by

$$K_{f,A,R}^{(m)} h = \sum_{\alpha\in\mathbb{F}_n^+} \sqrt{b_\alpha^{(m)}} e_\alpha \otimes R^{1/2} A_\alpha^* h, \qquad h \in \mathcal{H}. \tag{1.3}$$

Lemma 1.1. *The noncommutative Berezin kernel $K_{f,A,R}^{(m)}$ associated with a compatible quadruple (f, m, A, R) is a bounded operator and*

$$K_{f,A,R}^{(m)} A_i^* = (W_i^* \otimes I_\mathcal{R}) K_{f,A,R}^{(m)}, \qquad i = 1, \ldots, n,$$

where $\mathcal{R} := \overline{R^{1/2}(\mathcal{H})}$ *and* $(W_1,\dots,W_n)$ *is the universal model associated with the noncommutative domain* $\mathbf{D}_f^m$. *Moreover,*

$$\left(K_{f,A,R}^{(m)}\right)^* K_{f,A,R}^{(m)} = \sum_{k=0}^\infty \binom{k+m-1}{m-1} \Phi_{f,A}^k(R).$$

Proof. Since (f,m,A,R) is a compatible quadruple, $R\in B(\mathcal{H})$ is a positive operator such that

$$\sum_{k=0}^\infty \binom{k+m-1}{m-1} \Phi_{f,A}^k(R) \leq bI \tag{1.4}$$

for some constant $b>0$. Note that due to relations (1.1) and (1.3), we have

$$\begin{aligned}
\|K_{f,A,R}^{(m)}h\|^2 &= \sum_{\beta\in\mathbb{F}_n^+} b_\beta^{(m)} \left\langle A_\beta R A_\beta^* h, h\right\rangle = \langle Rh,h\rangle + \sum_{m=1}^\infty \sum_{|\beta|=m} \left\langle b_\beta^{(m)} A_\beta R A_\beta^* h, h\right\rangle\\
&= \langle Rh,h\rangle + \sum_{m=1}^\infty \sum_{|\beta|=m} \left\langle \left(\sum_{j=1}^{|\beta|} \binom{j+m-1}{m-1} \sum_{\substack{\gamma_1\cdots\gamma_j=\beta\\ |\gamma_1|\geq 1,\dots,|\gamma_j|\geq 1}} a_{\gamma_1}\cdots a_{\gamma_j}\right) A_{\gamma_1\cdots\gamma_j} R A_{\gamma_1\cdots\gamma_j}^* h, h\right\rangle\\
&= \langle Rh,h\rangle + \sum_{k=1}^\infty \left\langle \binom{k+m-1}{m-1} \Phi_{f,A}^k(R)h, h\right\rangle
\end{aligned}$$

for any $h\in\mathcal{H}$. Hence and due to relation (1.4), we deduce that $K_{f,A,R}^{(m)}$ is a well-defined bounded operator and

$$\left(K_{f,A,R}^{(m)}\right)^* K_{f,A,R}^{(m)} = \sum_{k=0}^\infty \binom{k+m-1}{m-1} \Phi_{f,A}^k(R).$$

On the other hand, due to relations (1.3) and (1.2), we have

$$\begin{aligned}
(W_i^*\otimes I_{\mathcal{R}})K_{f,A,R}^{(m)}h &= \sum_{\alpha\in\mathbb{F}_n^+} \sqrt{b_\alpha^{(m)}} W_i^* e_\alpha \otimes R^{1/2} A_\alpha^* h\\
&= \sum_{\gamma\in\mathbb{F}_n^+} \sqrt{b_{g_i\gamma}^{(m)}} W_i^* e_{g_i\gamma} \otimes R^{1/2} A_{g_i\gamma}^* h\\
&= \sum_{\gamma\in\mathbb{F}_n^+} \sqrt{b_\gamma^{(m)}} e_\gamma \otimes R^{1/2} A_\gamma^* A_i^* h\\
&= K_{f,A,R}^{(m)} A_i^* h
\end{aligned}$$

for any $h\in\mathcal{H}$. Hence,

$$K_{f,A,R}^{(m)} A_i^* = (W_i^*\otimes I_{\mathcal{R}})K_{f,A,R}^{(m)}, \qquad i=1,\dots,n,$$

and the proof is complete. □

Let $f := \sum_{|\alpha|\geq 1} a_\alpha X_\alpha$ be a positive regular free holomorphic function and let $W_1,\dots,W_n$ and $\Lambda_1,\dots\Lambda_n$ be the weighted left and right creation operators, respectively, associated with the noncommutative domain $\mathbf{D}_f^m$. Let $\mathcal{P}$ be a family of noncommutative polynomials and define the noncommutative variety $\mathcal{V}_{f,\mathcal{P}}^m$ whose representation on a Hilbert space $\mathcal{H}$ is

$$\mathcal{V}_{f,\mathcal{P}}^m(\mathcal{H}) := \left\{(X_1,\dots,X_n)\in \mathbf{D}_f^m(\mathcal{H}) :\ p(X_1,\dots,X_n)=0 \quad \text{for any} \quad p\in\mathcal{P}\right\}.$$

We associate with $\mathcal{V}_{f,\mathcal{P}}^m$ the operators $B_1,\dots,B_n$ defined as follows. Consider the subspaces

$$\mathcal{M}_{\mathcal{P}} := \overline{\text{span}}\{W_\alpha p(W_1,\dots,W_n)W_\beta(1) :\ p\in\mathcal{P}, \alpha,\beta\in\mathbb{F}_n^+\}$$

and $\mathcal{N}_{\mathcal{P}} := F^2(H_n)\ominus\mathcal{M}_{\mathcal{P}}$. Throughout this paper, unless otherwise specified, we assume that $\mathcal{N}_{\mathcal{P}}\neq\{0\}$. It is easy to see that $\mathcal{N}_{\mathcal{P}}$ is invariant under each operator $W_1^*,\dots,W_n^*$ and $\Lambda_1^*,\dots,\Lambda_n^*$. Define

$$B_i := P_{\mathcal{N}_{\mathcal{P}}} W_i|_{\mathcal{N}_{\mathcal{P}}} \quad \text{and} \quad C_i := P_{\mathcal{N}_{\mathcal{P}}} \Lambda_i|_{\mathcal{N}_{\mathcal{P}}}, \qquad i=1,\dots,n,$$

where $P_{\mathcal{N}_\mathcal{P}}$ is the orthogonal projection of $F^2(H_n)$ onto $\mathcal{N}_\mathcal{P}$.

The n-tuple of operators $B := (B_1, \ldots, B_n) \in \mathcal{V}^m_{f,\mathcal{P}}(\mathcal{N}_\mathcal{P})$ plays the role of universal model for the noncommutative variety $\mathcal{V}^m_{f,\mathcal{P}}$. The noncommutative variety algebra $\mathcal{A}_n(\mathcal{V}^m_{f,\mathcal{P}})$ is the norm-closed algebra generated by $B_1, \ldots, B_n$ and the identity, while the Hardy algebra $F^\infty_n(\mathcal{V}^m_{f,\mathcal{P}})$ is the w^*-version. More on these Hardy algebras associated with noncommutative varieties can be found in [28] and [25].

Let (f, m, A, R) be a compatible quadruple. Assume that the n-tuple $A := (A_1, \ldots, A_n) \in B(\mathcal{H})^n$ has, in addition, the property that

$$p(A_1, \ldots, A_n) = 0, \qquad p \in \mathcal{P}.$$

Under these conditions, the tuple $q := (f, m, A, R, \mathcal{P})$ is called compatible. We define the *(constrained) noncommutative Berezin kernel* associated with the tuple q to be the operator $K_q : \mathcal{H} \to \mathcal{N}_\mathcal{P} \otimes \overline{R^{1/2}(\mathcal{H})}$ given by

$$K_q := (P_{\mathcal{N}_\mathcal{P}} \otimes I_{\overline{R^{1/2}(\mathcal{H})}}) K^{(m)}_{f,A,R},$$

where $K^{(m)}_{f,A,R}$ is the Berezin kernel associated with the quadruple (f, m, A, R) and defined by relation (1.3).

Lemma 1.2. *Let K_q be the noncommutative Berezin kernel associated with a compatible tuple $q := (f, m, A, R, \mathcal{P})$. Then*

$$K_q A_i^* = (B_i^* \otimes I_\mathcal{R}) K_q, \qquad i = 1, \ldots, n,$$

where $\mathcal{R} := \overline{R^{1/2}(\mathcal{H})}$ and $(B_1, \ldots, B_n)$ is the universal model associated with the noncommutative variety $\mathcal{V}^m_{f,\mathcal{P}}$. Moreover,

$$K_q^* K_q = \sum_{k=0}^\infty \binom{k+m-1}{m-1} \Phi^k_{f,A}(R).$$

Proof. Using Lemma 1.1 and the fact that $p(A_1, \ldots, A_n) = 0$ for all $p \in \mathcal{P}$, we obtain

$$\left\langle K^{(m)}_{f,A,R} x, [W_\alpha p(W_1, \ldots, W_n) W_\beta(1)] \otimes y \right\rangle = \left\langle x, A_\alpha p(A_1, \ldots, A_n) A_\beta (K^{(m)}_{f,A,R})^* (1 \otimes y) \right\rangle = 0$$

for any $x \in \mathcal{H}$, $y \in \overline{R^{1/2}(\mathcal{H})}$, and $p \in \mathcal{P}$. Hence, we deduce that

$$\operatorname{range} K^{(m)}_{f,A,R} \subseteq \mathcal{N}_\mathcal{P} \otimes \overline{R^{1/2}(\mathcal{H})}. \tag{1.5}$$

Taking into account the definition of the constrained Berezin kernel $K_q : \mathcal{H} \to \mathcal{N}_\mathcal{P} \otimes \overline{R^{1/2}(\mathcal{H})}$, one can use Lemma 1.1 and relation (1.5) to complete the proof. □

We introduce now the *noncommutative Berezin transform* $\mathbf{B}_q$ associated with the compatible tuple $q := (f, m, A, R, \mathcal{P})$ to be the operator $\mathbf{B}_q : B(\mathcal{N}_\mathcal{P}) \to B(\mathcal{H})$ given by

$$\mathbf{B}_q[\chi] := K_q^* [\chi \otimes I_\mathcal{R}] K_q, \qquad \chi \in B(\mathcal{N}_\mathcal{P}).$$

where $\mathcal{R} := \overline{R^{1/2}(\mathcal{H})}$. This transform will play an important role in this paper. To justify the terminology, we shall consider the particular case when the n-tuple $A := (A_1, \ldots, A_n)$ has the joint spectral radius

$$r_f(A_1, \ldots, A_n) := \lim_{k \to \infty} \|\Phi^k_{f,A}(I)\|^{1/2k} < 1.$$

Then, as in the particular case considered in [25], one can show that

$$\langle \mathbf{B}_q[\chi] x, y \rangle = \left\langle \left(I - \sum_{|\alpha| \geq 1} a_{\tilde{\alpha}} C^*_\alpha \otimes A_{\tilde{\alpha}} \right)^{-m} (\chi \otimes R) \left(I - \sum_{|\alpha| \geq 1} a_{\tilde{\alpha}} C_\alpha \otimes A^*_{\tilde{\alpha}} \right)^{-m} (1 \otimes x), 1 \otimes y \right\rangle$$

for any $x, y \in \mathcal{H}$, where $C_i := P_{\mathcal{N}_\mathcal{P}} \Lambda_i|_{\mathcal{N}_\mathcal{P}}$ for $i = 1, \ldots, n$ and $\tilde{\alpha}$ is the reverse of $\alpha \in \mathbb{F}^+_n$. For the benefit of the reader, we present a sketch of the proof. First, one can show that

$$r\left(\sum_{|\alpha| \geq 1} a_{\tilde{\alpha}} C_\alpha \otimes A^*_{\tilde{\alpha}} \right) \leq r_f(A_1, \ldots, A_n) < 1,$$

where $r(Y)$ is the usual spectral radius of a bounded operator Y. Hence, the operator

$$\left(I-\sum_{|\alpha|\geq 1} a_{\widetilde{\alpha}} C_\alpha \otimes A_{\widetilde{\alpha}}^*\right)^{-1}=\sum_{k=0}^\infty \left(\sum_{|\alpha|\geq 1} a_{\widetilde{\alpha}} C_\alpha \otimes A_{\widetilde{\alpha}}^*\right)^k$$

is well-defined, where the convergence is in the operator norm topology. Consequently, using the definition of $\Lambda_1,\ldots,\Lambda_n$ and relation (1.3), we obtain

$$K_{f,T}^{(m)} h=(I_{F^2(H_n)}\otimes R^{1/2})\left(I-\sum_{|\alpha|\geq 1} a_{\widetilde{\alpha}}\Lambda_\alpha\otimes T_{\widetilde{\alpha}}^*\right)^{-m}(1\otimes h), \qquad h\in\mathcal{H}.$$

Combining the above-mentioned results with the fact that $K_q:=(P_{\mathcal{N}_\mathcal{P}}\otimes I_{\overline{R^{1/2}(\mathcal{H})}})K_{f,A,R}^{(m)}$, one can complete the proof of our assertion.

We remark that in the particular case when: $n=m=1$, $f=X$, $\mathcal{H}=\mathbb{C}$, $A=\lambda\in\mathbb{D}$, $R=I$, and $\mathcal{P}=\{0\}$, we recover the Berezin transform [2] of a bounded operator on the Hardy space $H^2(\mathbb{D})$, i.e.,

$$\mathbf{B}_\lambda[g]=(1-|\lambda|^2)\left\langle gk_\lambda,k_\lambda\right\rangle, \quad g\in B(H^2(\mathbb{D})),$$

where $k_\lambda(z):=(1-\overline{\lambda}z)^{-1}$ and $z,\lambda\in\mathbb{D}$.

The following technical lemma is a slight extension of Lemma 1.4 and 2.2 from [25], where the operator D was positive. In our extension, D is a self-adjoint operator and the condition (a) is new. However, since the proof is similar to those from [25], we shall omit it. A linear map $\varphi:B(\mathcal{H})\to B(\mathcal{H})$ is called power bounded if there exists a constant $M>0$ such that $\|\varphi^k\|\leq M$ for any $k\in\mathbb{N}$.

Lemma 1.3. *Let $\varphi:B(\mathcal{H})\to B(\mathcal{H})$ be a positive linear map and let $D\in B(\mathcal{H})$ be a self-adjoint operator and $m\in\mathbb{N}$. Then the following statements hold:*

(i) *If φ is power bounded, then*

$$(id-\varphi)^m(D)\geq 0 \quad \text{if and only if} \quad (id-\varphi)^s(D)\geq 0, \quad s=1,2,\ldots,m.$$

ii) *Under either one of the conditions:*
   (a) *$(id-\varphi)^s(D)\geq 0$ for any $s=1,\ldots,m$, or*
   (b) *φ is power bounded and $(id-\varphi)^m(D)\geq 0$,*
   the following limit exists and

$$\lim_{k\to\infty} k^d\left\langle \varphi^k(id-\varphi)^d(D)h,h\right\rangle=\begin{cases}\lim_{k\to\infty}\left\langle\varphi^k(D)h,h\right\rangle & \text{if } d=0\\ 0 & \text{if } d=1,2,\ldots,m-1\end{cases}$$

   for any $h\in\mathcal{H}$.

In what follows we also need the following result. For information on completely bounded (resp. positive) maps, we refer to [15] and [16].

Lemma 1.4. *Let $f:=\sum_{|\alpha|\geq 1} a_\alpha X_\alpha$ be a positive regular free holomorphic function and let $A:=(A_1,\ldots,A_n)\in B(\mathcal{H})^n$ be an n-tuple of operators such that $\sum_{|\alpha|\geq 1} a_\alpha A_\alpha A_\alpha^*$ is convergent in the weak operator topology. Then the map $\Phi_{f,A}:B(\mathcal{H})\to B(\mathcal{H})$, defined by*

$$\Phi_{f,A}(X)=\sum_{|\alpha|\geq 1} a_\alpha A_\alpha X A_\alpha^*, \qquad X\in B(\mathcal{H}),$$

where the convergence is in the weak operator topology, is a completely positive linear map which is WOT-continuous on bounded sets. Moreover, if $0<r<1$, then

$$\Phi_{f,A}(X)=\text{WOT-}\lim_{r\to 1}\Phi_{f,rA}(X), \qquad X\in B(\mathcal{H}).$$

Proof. Note that, for any $x, y \in \mathcal{H}$ and any finite subset $\Lambda \subset \{\alpha \in \mathbb{F}_n^+ : |\alpha| \geq 1\}$, we have

$$\begin{aligned}\sum_{\alpha\in\Lambda} |\langle a_\alpha A_\alpha X A_\alpha^* x, y\rangle| &\leq \|X\| \sum_{\alpha\in\Lambda} a_\alpha \|A_\alpha^* x\| \|A_\alpha^* y\| \\ &\leq \|X\| \left(\sum_{\alpha\in\Lambda} a_\alpha \|A_\alpha^* x\|^2\right)^{1/2} \left(\sum_{\alpha\in\Lambda} a_\alpha \|A_\alpha^* y\|^2\right)^{1/2}.\end{aligned}$$

Now, since $\sum_{|\alpha|\geq 1} a_\alpha A_\alpha A_\alpha^*$ is convergent in the weak operator topology it is easy to see that the series $\Phi_{f,A}(X) = \sum_{|\alpha|\geq 1} a_\alpha A_\alpha X A_\alpha^*$ convergence is in the weak operator topology. Moreover, the above-mentioned inequality is true for any subset Λ in $\{\alpha \in \mathbb{F}_n^+ : |\alpha| \geq 1\}$. In particular, we deduce that

$$|\langle \Phi_{f,A}(X)x, y\rangle| \leq \|X\| \langle \Phi_{f,A}(I)x, x\rangle^{1/2} \langle \Phi_{f,A}(I)y, y\rangle^{1/2}, \qquad x, y \in \mathcal{H}.$$

On the other hand, since the map $\Phi_{f,A}^{(k)}(X) := \sum_{1\leq|\alpha|\leq k} a_\alpha A_\alpha X A_\alpha^*$, $X \in B(\mathcal{H})$, is completely positive for each $k \in \mathbb{N}$ and $\Phi_{f,A}(X) = \text{WOT-}\lim_{k\to\infty} \Phi_{f,A}^{(k)}(X)$, we deduce that $\Phi_{f,A}$ is a completely positive map on $B(\mathcal{H})$. Since $\sum\limits_{|\alpha|\geq 1} a_\alpha A_\alpha A_\alpha^*$ is convergent in the weak operator topology, for any $\epsilon > 0$ and $x, y \in \mathcal{H}$, there is $N_0 \in \mathbb{N}$ such that

$$\sum_{|\alpha|>N_0} \langle a_\alpha A_\alpha A_\alpha^* x, x\rangle < \epsilon \quad \text{and} \quad \sum_{|\alpha|>N_0} \langle a_\alpha A_\alpha A_\alpha^* y, y\rangle < \epsilon.$$

Using the above-mentioned inequalities, we deduce that

$$\sum_{|\alpha|>N_0} |\langle a_\alpha A_\alpha X A_\alpha^* x, y\rangle| \leq \epsilon \|X\|.$$

Now, it is easy to see that $\Phi_{f,A}$ is WOT-continuous on bounded sets. On the other hand, we also have $\sum_{|\alpha|>N_0} \left|\left\langle a_\alpha r^{|\alpha|} A_\alpha X A_\alpha^* x, y\right\rangle\right| \leq \epsilon\|X\|$ for any $r \in [0, 1]$. This can be used to show that $\Phi_{f,A}(X) = \text{WOT-}\lim_{r\to 1} \Phi_{f,rA}(X)$ for any $X \in B(\mathcal{H})$. The proof is complete. □

We remark that Lemma 1.4 remains true if $\{a_\alpha\}_{|\alpha|\geq 1}$ is just a sequence of positive numbers and $A := (A_1, \ldots, A_n) \in B(\mathcal{H})^n$ is an n-tuple of operators such that $\sum_{|\alpha|\geq 1} a_\alpha A_\alpha A_\alpha^*$ is convergent in the weak operator topology.

We denote by $C(f, A)^+$ the cone of all positive operators $D \in B(\mathcal{H})$ such that

$$(id - \Phi_{f,A})^s(D) \geq 0 \quad \text{for} \quad s = 1, \ldots, m.$$

We denote by $C_{rad}(f, A)^+$ the set of all operators $D \in C(f, A)^+$ such that there is $\delta \in (0, 1)$ with the property that $D \in C(f, rA)^+$ for any $r \in (\delta, 1]$.

A few examples are necessary. Note that if $m = 1$ then we always have $C(f, A)^+ = C_{rad}(f, A)^+$. We remark that if $m \geq 2$ and $p = a_1 X_1 + \cdots + a_n X_n$, $a_i > 0$, then we also have $C(p, A)^+ = C_{rad}(p, A)^+$. Indeed, it is enough to see that if $0 < r \leq 1$, then

$$\begin{aligned}(id - \Phi_{p,rA})^k(D) &= \left[(id - \Phi_{p,A}) + (1 - r)\Phi_{p,A}\right]^k (D) \\ &= \sum_{j=0}^{k} \binom{k}{j} (1 - r)^{k-j} \Phi_{p,A}^{k-j} (id - \Phi_{p,A})^j (D)\end{aligned}$$

for any $k = 1, \ldots, m$. Since $(id - \Phi_{p,A})^j(D) \geq 0$ for $j = 1, \ldots, m$ and using the fact that $\Phi_{p,A}^j$ is a positive linear map, we deduce that $(id - \Phi_{p,rA})^k(D) \geq 0$ for $k = 1, \ldots, m$ and $r \in (0, 1]$, which proves our assertion. Note also that when $m \geq 1$ and q is any positive regular noncommutative polynomial so that, for each $s = 1, \ldots, m$, $(id - \Phi_{q,A})^s(D)$ is a positive invertible operator, then $D \in C_{rad}(q, A)^+$.

We say that $\mathbf{D}_f^m(\mathcal{H})$ is a *radial domain* if there exists $\delta \in (0, 1)$ such that $(rW_1, \ldots, rW_n) \in \mathbf{D}_f^m(F^2(H_n))$ for any $r \in (\delta, 1]$, where $(W_1, \ldots, W_n)$ is the universal model associated with $\mathbf{D}_f^m$. We remark that the notion of radial domain does not depend on the Hilbert space $\mathcal{H}$. Note that if $m = 1$, then $\mathbf{D}_f^1(\mathcal{H})$ is always a radial domain. This case was extensively studied in [28]. When $m \geq 2$, we point out the particular case $p := a_1 X_1 + \cdots + a_n X_n$, $a_i > 0$, when $\mathbf{D}_p^m(\mathcal{H})$ is also a radial domain.

Now, we are ready to show that, for radial domains $\mathbf{D}_f^m(\mathcal{H})$, the elements of the noncommutative cone $C_{rad}(f,A)^+$ are in one-to-one correspondence with the elements of a class of noncommutative Berezin transforms.

Theorem 1.5. *Let $\mathbf{D}_f^m(\mathcal{H})$ be a radial domain, where $f := \sum_{|\alpha|\geq 1} a_\alpha X_\alpha$ is a positive regular free holomorphic function and $m \geq 1$. Let $\mathcal{P}$ be a family of noncommutative homogeneous polynomials and let $B := (B_1,\ldots,B_n)$ be the universal model associated with the noncommutative variety $\mathcal{V}_{f,\mathcal{P}}^m$. If $A := (A_1,\ldots,A_n) \in B(\mathcal{H})^n$ is such that $\sum_{|\alpha|\geq 1} a_\alpha A_\alpha A_\alpha^*$ is SOT-convergent and $p(A_1,\ldots,A_n) = 0$, $p \in \mathcal{P}$, then there is a bijection*

$$\Gamma : CP(A,\mathcal{V}_{f,\mathcal{P}}^m) \to C_{rad}(f,A)^+, \qquad \Gamma(\varphi) := \varphi(I),$$

where $CP(A,\mathcal{V}_{f,\mathcal{P}}^m)$ is the set of all completely positive linear maps $\varphi : \mathcal{S}_{f,\mathcal{P}} \to B(\mathcal{H})$ such that

$$\varphi(B_\alpha B_\beta^*) = A_\alpha \varphi(I) A_\beta^*, \qquad \alpha,\beta \in \mathbb{F}_n^+,$$

where $\mathcal{S}_{f,\mathcal{P}} := \overline{\text{span}}\{B_\alpha B_\beta^* : \alpha,\beta \in \mathbb{F}_n^+\}$. Moreover, if $D \in C_{rad}(f,A)^+$, then $\Gamma^{-1}(D)$ coincides with the noncommutative Berezin transform associated with $q := (f,m,A,R,\mathcal{P})$ and defined by

$$\overline{\mathbf{B}}_q[\chi] := \lim_{r\to 1} K_{q_r}^*(\chi \otimes I) K_{q_r}, \qquad \chi \in \mathcal{S}_{f,\mathcal{P}},$$

where $q_r := (f,m,rA,R_r,\mathcal{P})$ and $R_r := (id - \Phi_{f,rA})^m(D)$, $r \in [0,1]$, and the limit exists in the operator norm topology.

Proof. We recall that the subspace $\mathcal{N}_\mathcal{P} \neq \{0\}$ is invariant under each operator $W_1^*,\ldots,W_n^*$ and $B_i := P_{\mathcal{N}_\mathcal{P}} W_i|_{\mathcal{N}_\mathcal{P}}$, $i = 1,\ldots,n$. Setting $B := (B_1,\ldots,B_n)$ and taking into account that $\Phi_{f,W}(I) \leq I$, we deduce that $\Phi_{f,B}(I) \leq I$ and, consequently, $\Phi_{f,rB}(I) = \sum_{k=1}^\infty \sum_{|\alpha|=k} a_\alpha r^{|\alpha|} B_\alpha B_\alpha^* \leq I$, where the convergence is in the operator norm topology. This implies $\Phi_{f,rB}(I) \in \mathcal{S}_{f,\mathcal{P}}$ for any $r \in [0,1)$. The fact that $\mathbf{D}_f^m$ is a radial domain implies $(rW_1,\ldots,rW_n) \in \mathbf{D}_f^m(F^2(H_n))$, $r \in (\delta,1)$, for some $\delta \in (0,1)$ and, consequently, $(id - \Phi_{f,rB})^s(I) \geq 0$ for $s = 1,\ldots,m$ and $r \in (\delta,1)$. Since

$$\Phi_{f,rB}^j(I) = \sum_{k=1}^\infty \sum_{|\alpha|=k} a_\alpha r^{|\alpha|} B_\alpha \Phi_{f,rB}^{j-1}(I) B_\alpha^*, \qquad j \in \mathbb{N},$$

and $\|\Phi_{f,rB}^k(I)\| \leq 1$ for any $k \in \mathbb{N}$, it is clear that $\Phi_{f,rB}^j(I) \in \mathcal{S}_{f,\mathcal{P}}$. Taking into account that

$$(id - \Phi_{f,rB})^s(I) = \sum_{j=0}^s (-1)^j \binom{s}{j} \Phi_{f,rB}^j(I), \qquad j \in \mathbb{N},$$

we deduce that $(id - \Phi_{f,rB})^s(I) \in \mathcal{S}_{f,\mathcal{P}}$ for $s = 1,\ldots,m$. Now, assume that $\varphi : \mathcal{S}_{f,\mathcal{P}} \to B(\mathcal{H})$ is a completely positive linear map such that

$$\varphi(B_\alpha B_\beta^*) = A_\alpha \varphi(I) A_\beta^*, \qquad \alpha,\beta \in \mathbb{F}_n^+.$$

Then, setting $D := \varphi(I)$, we deduce that $D \geq 0$ and

$$(id - \Phi_{f,rA})^s(D) = \varphi\left((id - \Phi_{f,rB})^s(I)\right) \geq 0, \qquad r \in (\delta,1),$$

for any $s = 1,\ldots,m$. Since the series $\sum\limits_{|\alpha|\geq 1} a_\alpha A_\alpha A_\alpha^*$ is SOT-convergent one can use Lemma 1.4 to deduce that $\Phi_{f,A}^k(D) = \text{WOT-}\lim_{r\to 1} \Phi_{f,rA}^k(D)$ for $k \in \mathbb{N}$ and, moreover,

$$(id - \Phi_{f,A})^s(D) = \text{WOT-}\lim_{r\to 1} (id - \Phi_{f,rA})^s(D) \geq 0$$

for any $s = 1,\ldots,m$. This shows that $D \in C_{rad}(f,A)^+$. To prove that Γ is one-to-one, let φ_1 and φ_2 be completely positive linear maps on $\mathcal{S}_{f,\mathcal{P}}$ such that $\varphi_j(B_\alpha B_\beta^*) = A_\alpha \varphi_i(I) A_\beta$, $\alpha,\beta \in \mathbb{F}_n^+$, and assume that $\Gamma(\varphi_1) = \Gamma(\varphi_2)$, i.e., $\varphi_1(I) = \varphi_2(I)$. Then we have $\varphi_1(B_\alpha B_\beta^*) = \varphi_2(B_\alpha B_\beta^*)$ for $\alpha,\beta \in \mathbb{F}_n^+$. Taking into account the continuity of φ_1 and φ_2 in the operator norm, we deduce that $\varphi_1 = \varphi_2$.

To prove surjectivity, fix $D \in C_{rad}(f,A)^+$. Then $D \in B(\mathcal{H})$ is a positive operator with the property that there is $\delta \in (0,1)$ such that $(id - \Phi_{f,rA})^s(D) \geq 0$ for any $s = 1,\ldots,m$ and $r \in (\delta,1)$. Since the set $\mathcal{P}$ consists of homogeneous noncommutative polynomials , we have $p(rA_1,\ldots,rA_n) = 0$ for any

$p \in \mathcal{P}$ and $r \in (\delta, 1)$. We show now that, for each $r \in (\delta, 1)$, the tuple $q_r := (f, m, rA, R_r, \mathcal{P})$, where $R_r := (id - \Phi_{f,rA})^m(D)$, is compatible. Indeed, we can use the equality

$$\binom{i+j}{j} - \binom{i+j-1}{j} = \binom{i+j-1}{j-1}, \qquad i, j \in \mathbb{N},$$

and Lemma 1.3 to obtain

$$\begin{aligned}\sum_{k=0}^{\infty} \binom{k+m-1}{m-1} \Phi_{f,rA}^k(R_r) &= D - \text{WOT-}\lim_{k\to\infty} \sum_{j=0}^{m-1} \binom{k+j}{j} \Phi_{f,rA}^{k+1}(id - \Phi_{f,rA})^j(D) \\ &= D - \text{WOT-}\lim_{k\to\infty} \Phi_{f,rA}^k(D).\end{aligned}$$

Since $\Phi_{f,rA}^k(D) \leq r^{2k}\Phi_{f,A}^k(D) \leq r^{2k}D$, we have $\text{WOT-}\lim_{k\to\infty} \Phi_{f,rA}^k(D) = 0$. Therefore, we deduce that

$$\sum_{k=0}^{\infty} \binom{k+m-1}{m-1} \Phi_{f,rA}^k(R_r) = D, \qquad r \in (\delta, 1). \tag{1.6}$$

According to Lemma 1.2, the constrained noncommutative Berezin kernel K_{q_r}, $r \in (\delta, 1)$, associated with the compatible tuple $q_r := (f, rA, R_r, \mathcal{P})$, has the property that

$$K_{q_r}(rA_i^*) = (B_i^* \otimes I_{\mathcal{H}})K_{q_r}, \qquad i = 1, \ldots, n, \tag{1.7}$$

where $(B_1, \ldots, B_n)$ is the n-tuple of constrained weighted left creation operators associated with the noncommutative variety $\mathcal{V}_{f,\mathcal{P}}^m$, and

$$K_{q_r}^* K_{q_r} = \sum_{k=0}^{\infty} \binom{k+m-1}{m-1} \Phi_{f,rA}^k(R_r), \qquad r \in (\delta, 1),$$

where $R_r := (id - \Phi_{f,rA})^m(D)$. Hence and using relation (1.6), we obtain

$$K_{q_r}^* K_{q_r} = D, \qquad r \in (\delta, 1). \tag{1.8}$$

For each $r \in (\delta, 1)$, define the operator $\mathbf{B}_{q_r} : \mathcal{S}_{f,\mathcal{P}} \to B(\mathcal{H})$ by setting

$$\mathbf{B}_{q_r}(\chi) := K_{q_r}^*(\chi \otimes I_{\mathcal{H}})K_{q_r}, \qquad \chi \in \mathcal{S}_{f,\mathcal{P}}. \tag{1.9}$$

Using relation (1.7) and (1.8), we have

$$K_{q_r}^*(B_\alpha B_\beta^* \otimes I)K_{q_r} = r^{|\alpha|+|\beta|} A_\alpha D A_\beta^*, \qquad \alpha, \beta \in \mathbb{F}_n^+, \ r \in (\delta, 1). \tag{1.10}$$

Hence, and using relations (1.8) and (1.9), we infer that $\mathbf{B}_{q_r}$ is a completely positive linear map with $\mathbf{B}_{q_r}(I) = D$ and $\|\mathbf{B}_{q_r}\| = \|D\|$ for $r \in (\delta, 1)$.

Now, we show that $\lim_{r\to 1} \mathbf{B}_{q_r}(\chi)$ exists in the operator norm topology for each $\chi \in \mathcal{S}_{f,\mathcal{P}}$. Given a polynomial $\varphi(B_1, \ldots, B_n) := \sum_{\alpha,\beta\in\mathbb{F}_n^+} a_{\alpha\beta} B_\alpha B_\beta^*$ in the operator system $\mathcal{S}_{f,\mathcal{P}}$, we define

$$\varphi_D(A_1, \ldots, A_n) := \sum_{\alpha,\beta\in\mathbb{F}_n^+} a_{\alpha\beta} A_\alpha D A_\beta^*.$$

The definition is correct since, according to relation (1.10), we have the following von Neumman type inequality

$$\|\varphi_D(A_1, \ldots, A_n)\| \leq \|D\|\|\varphi(B_1, \ldots, B_n)\|. \tag{1.11}$$

Now, fix $\chi \in \mathcal{S}_{f,\mathcal{P}}$ and let $\varphi^{(k)}(B_1, \ldots, B_n)$ be a sequence of polynomials in $\mathcal{S}_{f,\mathcal{P}}$ convergent to χ, in the operator norm topology. Define the operator

$$\chi_D(A_1, \ldots, A_n) := \lim_{k\to\infty} \varphi_D^{(k)}(A_1, \ldots, A_n). \tag{1.12}$$

Taking into account relation (1.11), it is clear that the operator $\chi_D(A_1, \ldots, A_n)$ is well-defined and

$$\|\chi_D(A_1, \ldots, A_n)\| \leq \|D\|\|\chi\|.$$

According to relation (1.10), we have

$$\|\varphi_D^{(k)}(rA_1, \ldots, rA_n)\| \leq \|D\|\|\varphi^{(k)}(B_1, \ldots, B_n)\|,$$

for any $r \in (\delta, 1)$. Taking into account that $\mathbf{B}_{q_r}$ is a bounded linear operator and using again relation (1.10), we deduce that

$$\lim_{k\to\infty} \varphi_D^{(k)}(rA_1, \ldots, rA_n) = \lim_{k\to\infty} K_{q_r}^*(\varphi^{(k)}(B_1, \ldots, B_n) \otimes I)K_{q_r} = \mathbf{B}_{q_r}[\chi], \tag{1.13}$$

for any $r \in (\delta, 1)$. Using relations (1.12), (1.13), the fact that $\|\chi - \varphi^{(k)}(B_1, \ldots, B_n)\| \to 0$ as $k \to \infty$, and

$$\lim_{r\to 1} \varphi_D^{(k)}(rA_1, \ldots, rA_n) = \varphi_D^{(k)}(A_1, \ldots, A_n),$$

we can deduce that

$$\lim_{r\to 1} \mathbf{B}_{q_r}[\chi] = \chi_D(A_1, \ldots, A_n)$$

in the norm topology. Indeed, note that

$$\begin{aligned}
&\|\chi_D(A_1, \ldots, A_n) - \mathbf{B}_{q_r}[\chi]\| \\
&\leq \|\chi_D(A_1, \ldots, A_n) - \varphi_D^{(k)}(A_1, \ldots, A_n)\| + \|\varphi_D^{(k)}(A_1, \ldots, A_n) - \mathbf{B}_{q_r}(\varphi^{(k)})\| \\
&\quad + \|\mathbf{B}_{q_r}(\varphi^{(k)}) - \mathbf{B}_{q_r}(\chi)\| \\
&\leq \|\chi - \varphi^{(k)}(B_1, \ldots, B_n)\|\|D\| + \|\varphi_D^{(k)}(A_1, \ldots, A_n) - \varphi_D^{(k)}(rA_1, \ldots, rA_n)\| \\
&\quad + \|\chi - \varphi^{(k)}(B_1, \ldots, B_n)\|\|D\|.
\end{aligned}$$

For any $r \in (\delta, 1)$, $\mathbf{B}_{q_r}$ is a completely positive linear map. Hence, and using relation (1.10), we infer that

$$\overline{\mathbf{B}}_q[\chi] := \lim_{r\to 1} K_{q_r}^*(\chi \otimes I)K_{q_r}, \qquad \chi \in \mathcal{S}_{f,\mathcal{P}},$$

is a completely positive map with $\overline{\mathbf{B}}_q(I) = D$ and $\overline{\mathbf{B}}_q(B_\alpha B_\beta^*) = A_\alpha \overline{\mathbf{B}}_q(I) A_\beta$, $\alpha, \beta \in \mathbb{F}_n^+$. The proof is complete. $\square$

The following result is an extension of the noncommutative von Neumann inequality (see [33], [19], [21], [25]).

Corollary 1.6. *Under the hypotheses of Theorem 1.5, if $D \in C_{rad}(f, A)^+$, then we have the following von Neumann type inequality:*

$$\left\| \sum_{\alpha,\beta\in\Lambda} A_\alpha D A_\beta^* \otimes C_{\alpha,\beta} \right\| \leq \|D\| \left\| \sum_{\alpha,\beta\in\Lambda} B_\alpha B_\beta^* \otimes C_{\alpha,\beta} \right\|$$

for any finite set $\Lambda \subset \mathbb{F}_n^+$ and $C_{\alpha,\beta} \in B(\mathcal{E})$, where $\mathcal{E}$ is a Hilbert space. If, in addition, D is an invertible operator, then the map $u : \mathcal{A}_n(\mathcal{V}_{f,\mathcal{P}}^m) \to B(\mathcal{H})$ defined by

$$u(p(B_1, \ldots, B_n)) := p(A_1, \ldots, A_n)$$

is completely bounded with $\|u\|_{cb} \leq \|D^{-1/2}\|\|D^{1/2}\|$.

Proof. Due to relation (1.10), we have

$$(K_{q_r}^* \otimes I_\mathcal{E})(B_\alpha B_\beta^* \otimes I \otimes C_{\alpha,\beta})(K_{q_r} \otimes I_\mathcal{E}) = r^{|\alpha|+|\beta|} A_\alpha D A_\beta^* \otimes C_{\alpha,\beta}, \qquad \alpha, \beta \in \mathbb{F}_n^+,\ r \in (\delta, 1).$$

Since $K_{q_r}^* K_{q_r} = D$ for $r \in (\delta, 1)$, one can easily deduce the von Neumann type inequality. To prove the second part, note that, if D is invertible, then the first part of this corollary implies

$$\begin{aligned}
\|p(A_1, \ldots, A_n)\|^2 &\leq \|D^{-1/2}\|^2 \|p(A_1, \ldots, A_n) D^{1/2}\|^2 \\
&= \|D^{-1/2}\|^2 \|p(A_1, \ldots, A_n) D p(A_1, \ldots, A_n)^*\| \\
&\leq \|D^{-1/2}\|^2 \|D\| \|p(B_1, \ldots, B_n) p(B_1, \ldots, B_n)^*\| \\
&= \|D^{-1/2}\|^2 \|D^{1/2}\|^2 \|p(B_1, \ldots, B_n)\|^2
\end{aligned}$$

for any noncommutative polynomial p. A similar result holds if we pass to matrices. Therefore, we deduce that u is completely bounded with $\|u\|_{cb} \leq \|D^{-1/2}\|\|D^{1/2}\|$. The proof is complete. $\square$

Example 1.7. (i) *When $m = 1$, $f = X_1 + \cdots + X_n$, and $D = I$, we obtain the noncommutative Poisson transform introduced in* [21] *(case $\mathcal{P} = \{0\}$) and* [24] *(case $\mathcal{P} \neq \{0\}$).*

(ii) *When* $m=1$, $f=X_1+\cdots+X_n$, $\mathcal{P}=\{0\}$, *and* $D\geq 0$ *such that* $\sum_{i=1}^n A_iDA_i^*\leq D$, *we obtain the noncommutative Poisson transform from* [22].

(iii) *When* $m\geq 1$, $D=I$, *and* f *is an arbitrary positive regular free holomorphic function, we obtain the noncommutative Berezin transforms associated with noncommutative domains* $\mathbf{D}_f^m$ *or noncommutative varieties* $\mathcal{V}_{f,\mathcal{P}}^m$, *which were studied in* [25] *and* [28].

2. Generalized noncommutative Berezin transforms and the cone $C_{pure}(f,A)^+$

In this section, we study the noncommutative cone $C_{pure}(f,A)^+$ of all pure solutions of the operator inequalities $(id-\Phi_{f,A})^s(X)\geq 0$, $s=1,\ldots,m$. When A is a pure n-tuple of operators in the noncommutative variety $\mathcal{V}_{f,\mathcal{P}}^1(\mathcal{H})$, we obtain a complete description of the noncommutative cone $C(f,A)^+$.

Let $A:=(A_1,\ldots,A_n)\in B(\mathcal{H})^n$ be such that $\sum_{|\alpha|\geq 1}a_\alpha A_\alpha A_\alpha^*$ is convergent in the weak operator topology and recall that

$$\Phi_{f,A}(X):=\sum_{|\alpha|\geq 1}a_\alpha A_\alpha XA_\alpha^*, \qquad X\in B(\mathcal{H}),$$

where the convergence is in the weak operator topology. We assume that $\Phi_{f,A}$ is power bounded. A self-adjoint operator $C\in B(\mathcal{H})$ is called pure solution of the inequality $(id-\Phi_{f,A})^m(X)\geq 0$ if

$$(id-\Phi_{f,A})^m(C)\geq 0 \quad \text{and} \quad \text{SOT-}\lim_{k\to\infty}\Phi_{f,A}^k(C)=0.$$

Note that since $\Phi_{f,A}$ is power bounded, Lemma 1.3 implies $\Phi_{f,A}(C)\leq C$. This can be used to show that a pure self-adjoint solution is always a positive operator. In what follows we present a *canonical decomposition* for the self-adjoint solutions of the operator inequality $(id-\Phi_{f,A})^m(X)\geq 0$.

Theorem 2.1. *Let* $f:=\sum_{|\alpha|\geq 1}a_\alpha X_\alpha$ *be a positive regular free holomorphic function and* $m\geq 1$. *Let* $A:=(A_1,\ldots,A_n)\in B(\mathcal{H})^n$ *be such that* $\sum\limits_{|\alpha|\geq 1}a_\alpha A_\alpha A_\alpha^*$ *is convergent in the weak operator topology and* $\Phi_{f,A}$ *is power bounded. If* $Y=Y^*\in B(\mathcal{H})$ *is such that* $(id-\Phi_{f,A})^m(Y)\geq 0$, *then there exist operators* $B,C\in B(\mathcal{H})$ *with the following properties:*

(i) $Y=B+C$;

(ii) $B=B^*$ *and* $\Phi_{f,A}(B)=B$;

(iii) $C\geq 0$, $(id-\Phi_{f,A})^m(C)\geq 0$, *and* $\text{SOT-}\lim_{k\to\infty}\Phi_{f,A}^k(C)=0$.

Moreover, the decomposition $Y=B+C$ *is unique with the above-mentioned properties and*

$$B=\text{SOT-}\lim_{k\to\infty}\Phi_{f,A}^k(Y)=\text{SOT-}\lim_{k\to\infty}\frac{1}{k}\sum_{j=0}^{k-1}\Phi_{f,A}^j(Y).$$

Proof. Let $Y=Y^*\in B(\mathcal{H})$ be such that $(id-\Phi_{f,A})^m(Y)\geq 0$. Since $\Phi_{f,A}$ is power bounded, Lemma 1.3 implies $\Phi_{f,A}(Y)\leq Y$. Consequently, the sequence of self-adjoint operators $\{\Phi_{f,A}^k(Y)\}_{k=0}^\infty$ is bounded and decreasing. Thus it converges strongly to a selfadjoint operator $B:=\text{SOT-}\lim\limits_{k\to\infty}\Phi_{f,A}^k(Y)$. Since $\Phi_{f,A}$ is a WOT-continuous map, we have $\Phi_{f,A}(B)=B$. Note that $C:=Y-B\geq 0$ satisfies the inequality $\Phi_{f,A}(C)\leq C$, and $(id-\Phi_{f,A})^m(C)=(id-\Phi_{f,A})^m(Y)\geq 0$. Moreover, $\Phi_{f,A}^k(C)\to 0$ strongly, as $k\to\infty$.

To prove the uniqueness of the decomposition, suppose $Y=B_1+C_1$, where B_1 and C_1 have the same properties as B and C, respectively. Then

$$B-B_1=\Phi_{f,A}^k(B-B_1)=\Phi_{f,A}^k(C_1-C), \qquad k\in\mathbb{N}.$$

Taking $k\to\infty$, we get $B=B_1$ and, consequently, $C=C_1$. Since $0\leq\Phi_{f,A}^k(C)\leq C$, $k\in\mathbb{N}$, and $\text{SOT}-\lim\limits_{k\to\infty}\Phi_{f,A}^k(C)=0$, a standard argument shows that $\text{SOT-}\lim_{k\to\infty}\frac{1}{k}\sum_{j=0}^{k-1}\Phi_{f,A}^j(C)=0$. On the other hand, since $Y=B+C$ and $\Phi_{f,A}(B)=B$, we infer that

$$\frac{1}{k}\sum_{j=0}^{k-1}\Phi_{f,A}^j(Y)=B+\frac{1}{k}\sum_{j=0}^{k-1}\Phi_{f,A}^j(C).$$

Hence, the result follows. The proof is complete. □

We denote by $C_{pure}(f,A)^+$ the set of all operators $D\in B(\mathcal{H})$ such that

$$(id-\Phi_{f,A})^s(D)\geq 0, \qquad s=1,\ldots,m,$$

and $\Phi_{f,A}^k(D)\to 0$ strongly, as $k\to\infty$. Note that such an operator D is always positive.

Theorem 2.2. *Let $f:=\sum_{|\alpha|\geq 1} a_\alpha X_\alpha$ be a positive regular free holomorphic function and $m\geq 1$. Let $\mathcal{P}$ be a family of noncommutative polynomials with $\mathcal{N}_\mathcal{P}\neq\{0\}$ and let $B:=(B_1,\ldots,B_n)$ be the universal model associated with the noncommutative variety $\mathcal{V}_{f,\mathcal{P}}^m$. If $A:=(A_1,\ldots,A_n)\in B(\mathcal{H})^n$ is such that $\sum_{|\alpha|\geq 1} a_\alpha A_\alpha A_\alpha^*$ is SOT-convergent and $p(A_1,\ldots,A_n)=0$, $p\in\mathcal{P}$, then there is a bijection*

$$\Gamma: CP^{w^*}(A,\mathcal{V}_{f,\mathcal{P}}^m)\to C_{pure}(f,A)^+, \qquad \Gamma(\varphi):=\varphi(1),$$

where $CP^{w^*}(A,\mathcal{V}_{f,\mathcal{P}}^m)$ is the set of all w^*-continuous completely positive linear maps $\varphi:\mathcal{S}_{f,\mathcal{P}}^{w^*}\to B(\mathcal{H})$ such that

$$\varphi(B_\alpha B_\beta^*)=A_\alpha\varphi(I)A_\beta^*, \qquad \alpha,\beta\in\mathbb{F}_n^+,$$

where $\mathcal{S}_{f,\mathcal{P}}^{w^*}:=\overline{\text{span}}^{w^*}\{B_\alpha B_\beta^*:\ \alpha,\beta\in\mathbb{F}_n^+\}$. In addition, if $D\in C_{pure}(f,A)^+$, then $\Gamma^{-1}(D)$ coincides with the noncommutative Berezin transform associated with $q:=(f,m,A,R,\mathcal{P})$ and defined by

$$\mathbf{B}_q[\chi]:=K_q^*(\chi\otimes I)K_q, \qquad \chi\in\mathcal{S}_{f,\mathcal{P}}^{w^*},$$

where $R:=(id-\Phi_{f,A})^m(D)$.

Moreover, an operator $D\in B(\mathcal{H})$ is in $C_{pure}(f,A)^+$ if and only if there is a Hilbert space $\mathcal{D}$ and an operator $K:\mathcal{H}\to\mathcal{N}_\mathcal{P}\otimes\mathcal{D}$ such that

$$D=K^*K \quad \text{and} \quad KA_i^*=(B_i^*\otimes I_\mathcal{D})K, \qquad i=1,\ldots,n.$$

Proof. Assume that $\varphi:\mathcal{S}_{f,\mathcal{P}}^{w^*}\to B(\mathcal{H})$ is a w^*-continuous completely positive linear map such that

$$\varphi(B_\alpha B_\beta^*)=A_\alpha\varphi(I)A_\beta^*, \qquad \alpha,\beta\in\mathbb{F}_n^+.$$

Then, setting $D:=\varphi(I)$ and taking into account that $\Phi_{f,B}=\sum_{|\alpha|\geq 1} a_\alpha B_\alpha B_\alpha^*$ is SOT convergent, we deduce that

$$(id-\Phi_{f,A})^s(D)=\varphi\left((id-\Phi_{f,B})^s(I)\right)\geq 0, \qquad s=1,\ldots,m.$$

On the other hand, recall that $\{\Phi_{f,B}^k(I)\}_{k=1}^\infty$ is a bounded decreasing sequence of positive operators which converges strongly to 0, as $k\to\infty$. Since $\Phi_{f,A}^k(D)=\varphi(\Phi_{f,B}^k(I))$ for all $k\in\mathbb{N}$, one can easily see that $\{\Phi_{f,A}^k(D)\}_{k=1}^\infty$ is a bounded decreasing sequence of positive operators which converges strongly, as $k\to\infty$. Taking into account that φ is continuous in the w^*-topology, which coincides with the weak operator topology on bounded sets, we deduce that $\Phi_{f,A}^k(D)\to 0$ strongly, as $k\to\infty$. Therefore, $D\in C_{pure}(f,A)^+$. To prove that Γ is one-to-one, let φ_1 and φ_2 be w^*-continuous completely positive linear maps on $\mathcal{S}_{f,\mathcal{P}}^{w^*}$ such that $\varphi_j(B_\alpha B_\beta^*)=A_\alpha\varphi_j(I)A_\beta$, $\alpha,\beta\in\mathbb{F}_n^+$, and assume that $\Gamma(\varphi_1)=\Gamma(\varphi_2)$, i.e., $\varphi_1(I)=\varphi_2(I)$. Then we have $\varphi_1(B_\alpha B_\beta^*)=\varphi_2(B_\alpha B_\beta^*)$ for $\alpha,\beta\in\mathbb{F}_n^+$. Since φ_1 and φ_2 are w^*-continuous, we deduce that $\varphi_1=\varphi_2$.

We prove now that Γ is a surjective map. Let $D\in C_{pure}(f,A)^+$ be fixed. According to Lemma 1.2, the constrained noncommutative Berezin kernel K_q associated with the compatible tuple $q:=(f,m,A,R,\mathcal{P})$, has the property that

$$K_qA_i^*=(B_i^*\otimes I_\mathcal{H})K_q, \quad i=1,\ldots,n, \tag{2.1}$$

where $(B_1,\ldots,B_n)$ is the universal model associated with the noncommutative variety $\mathcal{V}_{f,\mathcal{P}}^m$, and

$$K_q^*K_q=\sum_{k=0}^\infty \binom{k+m-1}{m-1}\Phi_{f,A}^k(R),$$

where $R:=(id-\Phi_{f,A})^m(D)$. As in the proof of Theorem 1.5, we can use Lemma 1.3 and the fact that $\text{WOT-}\lim_{k\to\infty}\Phi_{f,A}^k(D)=0$, to obtain

$$K_q^*K_q=\sum_{k=0}^\infty \binom{k+m-1}{m-1}\Phi_{f,A}^k(R)=D-\text{WOT-}\lim_{k\to\infty}\Phi_{f,A}^k(D)=D$$

Define the operator $\mathbf{B}_q : \mathcal{S}_{f,\mathcal{P}}^{w^*} \to B(\mathcal{H})$ by setting

$$\mathbf{B}_q(\chi) := K_q^*(\chi \otimes I_{\mathcal{H}})K_q, \qquad \chi \in \mathcal{S}_{f,\mathcal{P}}^{w^*}.$$

Now, due to relation (2.1) it is easy to see that

$$\mathbf{B}_q(B_\alpha B_\beta^*) = K_q^*(B_\alpha B_\beta^* \otimes I)K_q = A_\alpha D A_\beta^*, \qquad \alpha, \beta \in \mathbb{F}_n^+.$$

Consequently, $\mathbf{B}_q \in CP^{w^*}(A, \mathcal{V}_{f,\mathcal{P}}^m)$ has the required properties.

To prove the last part of the theorem, note that the direct implication follows if we take K to be the noncommutative Berezin kernel K_q. To prove the converse, assume that there is a Hilbert space $\mathcal{D}$ and an operator $K : \mathcal{H} \to \mathcal{N}_{\mathcal{P}} \otimes \mathcal{D}$ such that

$$D = K^*K \quad \text{and} \quad KA_i^* = (B_i^* \otimes I_{\mathcal{D}})K, \qquad i = 1, \dots, n.$$

Then

$$(id - \Phi_{f,A})^s(D) = K^* \left[(id - \Phi_{f,B})^s(I) \otimes I_{\mathcal{D}}\right] K \geq 0, \qquad s = 1, \dots, m.$$

Since $\Phi_{f,A}^k(D) = K^*[\Phi_{f,B}^k(I) \otimes I_{\mathcal{D}}]K$, $\|\Phi_{f,B}^k(I)\| \leq 1$, and $\Phi_{f,B}^k(I) \to 0$ strongly, as $k \to 0$, we deduce that $D \in C_{pure}(f, A)^+$. The proof is complete. □

We remark that, in Theorem 2.2, the set $\mathcal{P}$ is of arbitrary noncommutative polynomials with $\mathcal{N}_{\mathcal{P}} \neq \{0\}$, while, in Theorem 1.5, $\mathcal{P}$ consists of homogeneous polynomials.

The proof of the next result is similar to that of Corollary 1.6, so we shall omit it.

Corollary 2.3. *Under the hypotheses of Theorem 2.2, if $D \in C_{pure}(f, A)^+$, then we have the following von Neumann type inequality:*

$$\left\| \sum_{\alpha,\beta \in \Lambda} A_\alpha D A_\beta^* \otimes C_{\alpha,\beta} \right\| \leq \|D\| \left\| \sum_{\alpha,\beta \in \Lambda} B_\alpha B_\beta^* \otimes C_{\alpha,\beta} \right\|$$

for any finite set $\Lambda \subset \mathbb{F}_n^+$ and $C_{\alpha,\beta} \in B(\mathcal{E})$, where $\mathcal{E}$ is a Hilbert space.

If, in addition, D is an invertible operator, then the polynomial calculus $p(B_1, \dots, B_n) \mapsto p(A_1, \dots, A_n)$ extends to a completely bounded map $u : F_n^\infty(\mathcal{V}_{f,\mathcal{P}}^m) \to B(\mathcal{H})$ by setting

$$u(\varphi) := K_q^*[\varphi \otimes I_{\mathcal{H}}]K_q D^{-1}, \qquad \varphi \in F_n^\infty(\mathcal{V}_{f,\mathcal{P}}^m),$$

where K_q is the noncommutative Berezin kernel associated with the compatible tuple $q := (f, m, A, R, \mathcal{P})$ and $R := (id - \Phi_{f,A})^m(D)$. Moreover, $\|u\|_{cb} \leq \|D^{-1/2}\|\|D^{1/2}\|$.

Theorem 2.4. *Let $f := \sum_{|\alpha| \geq 1} a_\alpha X_\alpha$ be a positive regular free holomorphic function and $m \geq 1$. Let $\mathcal{P}$ be a family of noncommutative polynomials and let $A := (A_1, \dots, A_n) \in B(\mathcal{H})^n$ be such that $\sum_{|\alpha| \geq 1} a_\alpha A_\alpha A_\alpha^*$ is SOT-convergent and $p(A_1, \dots, A_n) = 0$ for $p \in \mathcal{P}$. Then a positive operator $G \in B(\mathcal{H})$ is in $C(f, A)^+$ if and only if there exists an n-tuple $T := (T_1, \dots, T_n) \in \mathcal{V}_{f,\mathcal{P}}^m(\mathcal{H})$ such that*

$$A_i G^{1/2} = G^{1/2} T_i, \qquad i = 1, \dots, n.$$

In addition, $G \in C_{pure}(f, A)^+$ if and only if $I_{\mathcal{H}} \in C_{pure}(f, T)^+$.

Proof. First, assume that $T := (T_1, \dots, T_n) \in \mathcal{V}_{f,\mathcal{P}}^m(\mathcal{H})$ satisfies $A_i G^{1/2} = G^{1/2} T_i$, for any $i = 1, \dots, n$. Then we have

$$(id - \Phi_{f,A})^s(G) = G^{1/2} \left[(id - \Phi_{f,T})^s(I)\right] G^{1/2} \geq 0, \qquad s = 1, \dots, m.$$

Taking into account that $\Phi_{f,A}^k(G) = G^{1/2} \Phi_{f,T}^k(I) G^{1/2}$, $k \in \mathbb{N}$, it is clear that if $\Phi_{f,T}^k(I) \to 0$ strongly, as $k \to \infty$, then $G \in C_{pure}(f, A)^+$.

To prove the converse, assume that $G \in B(\mathcal{H})$ is in $C(f, A)^+$. Since

$$\sum_{|\alpha| \geq 1} \|G^{1/2} \sqrt{a_\alpha} A_\alpha^* x\|^2 = \langle \Phi_{f,A}(G)x, x \rangle \leq \|G^{1/2} x\|^2$$

for any $x \in \mathcal{H}$, we deduce that $a_{g_i}\|G^{1/2}A_i^* x\|^2 \le \|G^{1/2}x\|^2$, for any $x \in \mathcal{H}$. Recall that $a_{g_i} \neq 0$, so we can define the operator $\Lambda_i : G^{1/2}(\mathcal{H}) \to G^{1/2}(\mathcal{H})$ by setting

$$\Lambda_i G^{1/2}x := G^{1/2}A_i^* x, \qquad x \in \mathcal{H}, \tag{2.2}$$

for $i = 1, \ldots, n$. It is obvious that Λ_i can be extended to a bounded operator (also denoted by Λ_i) on the subspace $\mathcal{M} := \overline{G^{1/2}(\mathcal{H})}$. Set $M_i := \Lambda_i^*$, $i = 1, \ldots, n$, and note that

$$G^{1/2}\left[(id - \Phi_{f,M})^s(I_{\mathcal{M}})\right]G^{1/2} = (id - \Phi_{f,A})^s(G) \ge 0, \qquad s = 1, \ldots, m.$$

An approximation argument shows that

$$(id - \Phi_{f,M})^s(I_{\mathcal{M}}) \ge 0, \qquad s = 1, \ldots, m.$$

Define $T_i := M_i \oplus 0$, $i = 1, \ldots, n$, with respect to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, and note that $(id - \Phi_{f,T})^s(I) \ge 0$, $s = 1, \ldots, m$. Due to relation (2.2), if $p \in \mathcal{P}$, then we have

$$p(M_1, \ldots, M_n)^* G^{1/2} = G^{1/2}p(A_1, \ldots, A_n)^* = 0.$$

Hence, $p(M_1, \ldots, M_n) = 0$ and, consequently, $p(T_1, \ldots, T_n) = 0$ for all $p \in \mathcal{P}$. Therefore, $(T_1, \ldots, T_n) \in \mathcal{V}_{f,\mathcal{P}}^m(\mathcal{H})$ and $A_i G^{1/2} = G^{1/2}T_i$, $i = 1, \ldots, n$.

Assume now that $G \in C_{pure}(f, A)^+$, i.e., $\Phi_{f,A}^k(G) \to 0$ strongly, as $k \to \infty$. Since

$$\left\langle \Phi_{f,T}^k(I)G^{1/2}x, G^{1/2}x\right\rangle = \left\langle \Phi_{f,A}^k(G)x, x\right\rangle, \qquad x \in \mathcal{H},$$

we have SOT-$\lim_{k\to\infty} \Phi_{f,T}^k(I)y = 0$ for any $y \in$ range $G^{1/2}$. Taking into account that $\|\Phi_{f,T}^k(I)\| \le 1$, $k \in \mathbb{N}$, an approximation argument shows that SOT-$\lim_{k\to\infty} \Phi_{f,T}^k(I)y = 0$ for any $y \in \overline{G^{1/2}(\mathcal{H})}$. On the other hand, we have $\Phi_{f,T}^k(I)z = 0$ for any $z \in \mathcal{M}^\perp$. Consequently, $I_{\mathcal{H}} \in C_{pure}(f, T)^+$. This completes the proof. $\square$

In what follows we consider the case when $m = 1$. Let $f := \sum_{|\alpha|\ge 1} a_\alpha X_\alpha$ be a positive regular free holomorphic function and let $\mathcal{P}$ be a family of noncommutative polynomials such that $\mathcal{N}_{\mathcal{P}} \neq \{0\}$. We have

$$\mathbf{D}_f^1(\mathcal{H}) := \{(X_1, \ldots, X_n) \in B(\mathcal{H})^n : \sum_{|\alpha|\ge 1} a_\alpha X_\alpha X_\alpha^* \le I\}.$$

Let $B := (B_1, \ldots, B_n)$ be the universal model associated with the noncommutative variety $\mathcal{V}_{f,\mathcal{P}}^1$. We introduced in [28] the noncommutative Hardy algebra $F_n^\infty(\mathcal{V}_{f,\mathcal{P}}^1)$ to be the w^*-closed algebra generated by $B_1, \ldots, B_n$ and the identity. We also showed that $F_n^\infty(\mathcal{V}_{f,\mathcal{P}}^1) = P_{\mathcal{N}_{\mathcal{P}}} F_n^\infty(\mathbf{D}_f^1)|_{\mathcal{N}_{\mathcal{P}}}$. Similar results hold for $R_n^\infty(\mathcal{V}_{f,\mathcal{P}}^1)$, the w^*-closed algebra generated by $C_1, \ldots, C_n$ and the identity, where $C_i := P_{\mathcal{N}_{\mathcal{P}}}\Lambda_i|_{\mathcal{N}_{\mathcal{P}}}$, and $\Lambda_1, \ldots, \Lambda_n$ are the weighted right creation operators associated with $\mathbf{D}_f^1$ (see Section 1). Moreover, we proved that

$$F_n^\infty(\mathcal{V}_{f,\mathcal{P}}^1)' = R_n^\infty(\mathcal{V}_{f,\mathcal{P}}^1) \quad \text{and} \quad R_n^\infty(\mathcal{V}_{f,\mathcal{P}}^1)' = F_n^\infty(\mathcal{V}_{f,\mathcal{P}}^1),$$

where $'$ stands for the commutant. An operator $M \in B(\mathcal{N}_{\mathcal{P}} \otimes \mathcal{K}, \mathcal{N}_{\mathcal{P}} \otimes \mathcal{K}')$ is called multi-analytic with respect to the constrained weighted shifts $B_1, \ldots, B_n$ if

$$M(B_i \otimes I_{\mathcal{K}}) = (B_i \otimes I_{\mathcal{K}'})M, \qquad i = 1, \ldots, n.$$

According to [28], the set of all multi-analytic operators with respect to $B_1, \ldots, B_n$ coincides with

$$R_n^\infty(\mathcal{V}_{f,\mathcal{P}}^1)\bar{\otimes}B(\mathcal{K}, \mathcal{K}') = P_{\mathcal{N}_{\mathcal{P}}\otimes\mathcal{K}'}[R_n^\infty(\mathbf{D}_f^1)\bar{\otimes}B(\mathcal{K}, \mathcal{K}')]|_{\mathcal{N}_{\mathcal{P}}\otimes\mathcal{K}},$$

and a similar result holds for the Hardy algebra $F_n^\infty(\mathcal{V}_{f,\mathcal{P}}^1)$. For more information on multi-analytic operators, we refer the reader to [20] and [28].

Theorem 2.5. *Let $\mathcal{P}$ be a family of noncommutative polynomials with $\mathcal{N}_{\mathcal{P}} \neq \{0\}$ and let $B := (B_1, \ldots, B_n)$ be the universal model associated with the noncommutative variety $\mathcal{V}_{f,\mathcal{P}}^1$, where $f := \sum_{|\alpha|\ge 1} a_\alpha X_\alpha$ is a positive regular free holomorphic function. If $T := (T_1, \ldots, T_n)$ is a pure n-tuple of operators in the noncommutative variety $\mathcal{V}_{f,\mathcal{P}}^1(\mathcal{H})$, then*

$$C(f, T)^+ = C_{pure}(f, T)^+$$

and any operator in $C(f,T)^+$ *has the form* $G = P_\mathcal{H}\Psi\Psi^*|_\mathcal{H}$, *where* Ψ *is a multi-analytic operator with respect to* $B_1,\dots,B_n$.

Proof. Assume that $T := (T_1,\dots,T_n)$ is a pure n-tuple of operators in the noncommutative variety $\mathcal{V}^1_{f,\mathcal{P}}(\mathcal{H})$, i.e., $\Phi^k_{f,T}(I) \to 0$ strongly, as $k\to\infty$. If $G \in C(f,T)^+$, then $G \geq 0$ and $\Phi_{f,T}(G) \leq G$. Since

$$0 \leq \Phi^k_{f,T}(G) \leq \|G\|\,\Phi^k_{f,T}(I), \qquad k = 1,2,\dots,$$

we infer that $G \in C_{pure}(f,T)^+$. Consequently, we have $C(f,T)^+ = C_{pure}(f,T)^+$. Now, fix an operator $G \in C_{pure}(f,T)^+$. Due to Theorem 2.4, we find $D_i \in B(\mathcal{H})$ satisfying

$$T_iG^{1/2} = G^{1/2}D_i, \qquad i = 1,\dots,n,$$

where $(D_1,\dots,D_n) \in \mathcal{V}^1_{f,\mathcal{P}}(\mathcal{H})$ and $\Phi^k_{f,D}(I) \to 0$ strongly, as $k \to 0$. According to Theorem 3.20 from [28], there is a Hilbert space $\mathcal{M}_1$ so that $(B_1\otimes I_{\mathcal{M}_1},\dots,B_n\otimes I_{\mathcal{M}_1})$ is a dilation of $(T_1,\dots,T_n)$ on the Hilbert space $\mathcal{K}_1 := \mathcal{N}_\mathcal{P}\otimes\mathcal{M}_1 \supseteq \mathcal{H}$, i.e., $T_i = P_\mathcal{H}(B_i\otimes I_{\mathcal{M}_1})|_\mathcal{H}$, $i=1,\dots,n$, and $\mathcal{H}$ is invariant under each operator $B_i^*\otimes I_{\mathcal{M}_1}$. Similarly, let $(B_1\otimes I_{\mathcal{M}_2},\dots,B_n\otimes I_{\mathcal{M}_2})$ be a dilation of $(D_1,\dots,D_n)$ on a Hilbert space $\mathcal{K}_2 := \mathcal{N}_\mathcal{P}\otimes\mathcal{M}_2 \supseteq \mathcal{H}$. According to the noncommutative commutant lifting theorem from [28] (see Theorem 4.2), there exists an operator $\widehat{G} : \mathcal{K}_2 \to \mathcal{K}_1$ such that $\widehat{G}^*(\mathcal{H}) \subset \mathcal{H}$, $\widehat{G}^*|_\mathcal{H} = G^{1/2}$, $\|\widehat{G}\| = \|G^{1/2}\|$, and

$$\widehat{G}^*(B_i^*\otimes I_{\mathcal{M}_1}) = (B_i^*\otimes I_{\mathcal{M}_2})\widehat{G}^*, \qquad i=1,\dots,n.$$

It is easy to see that

$$\Phi_{f,B\otimes I_{\mathcal{M}_1}}(\widehat{G}\widehat{G}^*) = \widehat{G}\Phi_{f,B\otimes I_{\mathcal{M}_2}}(I)\widehat{G}^* \leq \widehat{G}\widehat{G}^*.$$

Setting $Q := \widehat{G}\widehat{G}^*$, we have $\|Q\| = \|G\|$, and

$$G = P_\mathcal{H}\widehat{G}|_\mathcal{H}G^{1/2} = P_\mathcal{H}\widehat{G}\widehat{G}^*|_\mathcal{H} = P_\mathcal{H}Q|_\mathcal{H}.$$

Note also that

$$\Phi^k_{f,B\otimes I_{\mathcal{M}_1}}(\widehat{G}\widehat{G}^*) = \widehat{G}\Phi^k_{f,B\otimes I_{\mathcal{M}_2}}(I)\widehat{G}^*, \qquad k\in\mathbb{N}.$$

Since $\Phi^k_{f,B\otimes I_{\mathcal{M}_2}}(I) \to 0$ strongly, as $k\to\infty$, we deduce that $\Phi^k_{f,B\otimes I_{\mathcal{M}_1}}(\widehat{G}\widehat{G}^*) \to 0$ strongly. Therefore, $Q \in C_{pure}(f, B\otimes I_{\mathcal{M}_1})^+$ and $G = P_\mathcal{H}Q|_\mathcal{H}$.

Conversely, if $Q \in C_{pure}(f, B\otimes I_{\mathcal{M}_1})^+$, then

$$\begin{aligned}\Phi_{f,T}(P_\mathcal{H}Q|_\mathcal{H}) &= \sum_{|\alpha|\geq 1} a_\alpha T_\alpha(P_\mathcal{H}Q|_\mathcal{H})T_\alpha^* \\ &= P_\mathcal{H}[\Phi_{f,W\otimes I_{\mathcal{M}_1}}(P_\mathcal{H}Q|_\mathcal{H})]|_\mathcal{H} \\ &\leq P_\mathcal{H}[\Phi_{f,W\otimes I_{\mathcal{M}_1}}(Q)]|_\mathcal{H} \\ &\leq P_\mathcal{H}Q|_\mathcal{H}.\end{aligned}$$

On the other hand, since

$$0 \leq \Phi^k_{f,T}(P_\mathcal{H}Q|_\mathcal{H}) \leq P_\mathcal{H}\Phi^k_{f,B\otimes I_{\mathcal{M}_1}}(Q)|_\mathcal{H} \to 0, \quad \text{as } k\to\infty,$$

it is clear that $G := P_\mathcal{H}Q|_\mathcal{H}$ is in $C_{pure}(f,T)^+$. We have proved that

$$C_{pure}(f,T)^+ = P_\mathcal{H}\left[C_{pure}(f, B\otimes I_{\mathcal{M}_1})^+\right]|_\mathcal{H}.$$

Now, we determine the set $C_{pure}(f, B\otimes I_{\mathcal{M}_1})^+$. To this end, let $Q \in C_{pure}(f, B\otimes I_{\mathcal{M}_1})^+$. According to Theorem 2.2, $Q \in C_{pure}(f, B\otimes I_{\mathcal{M}_1})^+$ if and only if there is a Hilbert space $\mathcal{D}$ and an operator $K : \mathcal{N}_\mathcal{P}\otimes\mathcal{M}_1 \to \mathcal{N}_\mathcal{P}\otimes\mathcal{D}$ such that $Q = K^*K$ and

$$(B_i\otimes I_{\mathcal{M}_1})K^* = K^*(B_i\otimes I_\mathcal{D}), \qquad i=1,\dots,n,$$

i.e., K^* is a multi-analytic operator with respect to $B_1,\dots,B_n$. The proof is complete. $\square$

3. Joint similarity to operators in noncommutative varieties

In this section we provide necessary and sufficient conditions for an n-tuple $A := (A_1, \dots, A_n) \in B(\mathcal{H})^n$ to be jointly similar to an n-tuple $T := (T_1, \dots, T_n)$ satisfying one of the following properties:

(i) $T \in \mathcal{V}^m_{f,\mathcal{P}}(\mathcal{H})$;

(ii) $T \in \left\{X \in \mathcal{V}^m_{f,\mathcal{P}}(\mathcal{H})) : \ (id - \Phi_{f,X})^m(I) = 0\right\}$;

(iii) $T \in \left\{X \in \mathcal{V}^m_{f,\mathcal{P}}(\mathcal{H})) : \ (id - \Phi_{f,X})^m(I) > 0\right\}$;

(iv) T is a pure n-tuple in $\mathcal{V}^m_{f,\mathcal{P}}(\mathcal{H})$, i.e., $\Phi^k_{f,T}(I) \to 0$ strongly, as $k \to \infty$,

where $\mathcal{P}$ is a set of noncommutative polynomials. We show that these similarities are strongly related to the existence of invertible positive solutions of the operator inequality $(id - \Phi_{f,A})^m(Y) \geq 0$ or equation $(id - \Phi_{f,A})^m(Y) = 0$. Several classical results concerning the similarity to contractions have analogues in our multivariable setting.

Let $f = \sum_{|\alpha|\geq 1} a_\alpha X_\alpha$ be a positive regular free holomorphic. For any n-tuple of operators $A := (A_1, \dots, A_n) \in B(\mathcal{H})^n$ such that $\sum_{|\alpha|\geq 1} a_\alpha A_\alpha A_\alpha^*$ is convergent in the weak operator topology, define the joint spectral radius with respect to the noncommutative domain $\mathbf{D}^m_f$ by setting

$$r_f(A_1, \dots, A_n) := \lim_{k\to\infty} \|\Phi^k_{f,A}(I)\|^{1/2k}.$$

In the particular case when $f := X_1 + \cdots + X_n$, we obtain the usual definition of the joint operator radius for n-tuples of operators.

Our first result provides necessary conditions for joint similarity to n-tuples of operators in noncommutative varieties $\mathcal{V}^m_{f,\mathcal{P}}(\mathcal{H})$.

Proposition 3.1. *Let $f := \sum_{|\alpha|\geq 1} a_\alpha X_\alpha$ be a positive regular free holomorphic function and let $T := (T_1, \dots, T_n) \in B(\mathcal{H})^n$ and $A := (A_1, \dots, A_n) \in B(\mathcal{K})^n$ be two n-tuples of operators which are jointly similar, i.e., there exists an invertible operator $Y : \mathcal{H} \to \mathcal{K}$ such that*

$$A_i = Y T_i Y^{-1}, \qquad i = 1, \dots, n.$$

If $\mathcal{P}$ is a family of noncommutative polynomials and $T \in \mathcal{V}^m_{f,\mathcal{P}}(\mathcal{H})$, then the following statements hold:

(i) *$\sum_{|\alpha|\geq 1} a_\alpha A_\alpha A_\alpha^*$ is convergent in the weak operator topology;*

(ii) *$\Phi_{f,A}$ is a power bounded completely positive linear map;*

(iii) *$r_f(A_1, \dots, A_n) \leq 1$;*

(iv) *$p(A_1, \dots, A_n) = 0$ for all $p \in \mathcal{P}$;*

(v) *if $\Phi^k_{f,T}(I) \to 0$ strongly, as $k \to \infty$, then $\Phi^k_{f,A}(I) \to 0$ strongly.*

Proof. Note that

$$\begin{aligned}\sum_{1\leq|\alpha|\leq k} a_\alpha A_\alpha A_\alpha^* &= \sum_{1\leq|\alpha|\leq k} a_\alpha Y T_\alpha Y^{-1}(Y^{-1})^* T_\alpha^* Y^* \\ &\leq \|Y^{-1}\|^2\, Y \left(\sum_{1\leq|\alpha|\leq k} a_\alpha T_\alpha T_\alpha^*\right) Y^*\end{aligned}$$

for any $k \in \mathbb{N}$. Since $T := (T_1, \dots, T_n) \in \mathcal{V}^m_{f,\mathcal{P}}(\mathcal{H})$, the series $\sum_{|\alpha|\geq 1} a_\alpha T_\alpha T_\alpha^*$ is convergent in the weak operator topology and $p(T_1, \dots, T_n) = 0$ for all $p \in \mathcal{P}$. Now, due to inequality above, it is easy to see that item (i) holds and

$$\Phi_{f,A}(I) \leq \|Y^{-1}\|^2\, Y \Phi_{f,T}(I) Y^* \leq \|Y^{-1}\|^2 \|Y\|^2 I.$$

According to Lemma 1.4, $\Phi_{f,A}$ is a completely positive map. As above, one can also show that

$$\Phi^k_{f,A}(I) \leq \|Y^{-1}\|^2\, Y \Phi^k_{f,T}(I) Y^* \leq \|Y^{-1}\|^2 \|Y\|^2 I, \qquad k \in \mathbb{N},$$

which proves item (ii) and implies items (iii) and (v). Since item (iv) is obvious, the proof is complete. $\square$

We recall that $C(f,A)^+$ is the cone of all positive operators $D\in B(\mathcal{H})$ such that $(id-\Phi_{f,A})^s(D)\geq 0$ for $s=1,\ldots,m$. Now, we are ready to provide necessary and sufficient conditions for the joint similarity to parts of the adjoints of the universal model $(B_1,\ldots,B_n)$ associated with the noncommutative variety $\mathcal{V}^m_{f,\mathcal{P}}$.

Theorem 3.2. *Let $m\geq 1$, $f:=\sum_{|\alpha|\geq 1} a_\alpha X_\alpha$ be a positive regular free holomorphic function and let $\mathcal{P}$ be a family of noncommutative polynomials with $\mathcal{N}_\mathcal{P}\neq 0$. If $A:=(A_1,\ldots,A_n)\in B(\mathcal{H})^n$ is such that $\sum_{|\alpha|\geq 1} a_\alpha A_\alpha A_\alpha^*$ is convergent in the weak operator topology and $p(A_1,\ldots,A_n)=0,\quad p\in\mathcal{P}$, then the following statements are equivalent.*

(i) *There exists an invertible operator $Y:\mathcal{H}\to\mathcal{G}$ such that*
$$A_i^*=Y^{-1}[(B_i^*\otimes I_\mathcal{H})|_\mathcal{G}]Y,\qquad i=1,\ldots,n,$$
where $\mathcal{G}\subseteq \mathcal{N}_\mathcal{P}\otimes\mathcal{H}$ is an invariant subspace under each operator $B_i^*\otimes I_\mathcal{H}$ and $(B_1,\ldots,B_n)$ is the universal model associated with the noncommutative variety $\mathcal{V}^m_{f,\mathcal{P}}$.

(ii) *There is an invertible operator $Q\in C(f,A)^+$ such that $\Phi^k_{f,A}(Q)\to 0$ strongly, as $k\to\infty$.*

(iii) *There exist constants $0<a\leq b$ and a positive operator $R\in B(\mathcal{H})$ such that*
$$aI\leq \sum_{k=0}^\infty \binom{k+m-1}{m-1}\Phi^k_{f,A}(R)\leq bI.$$

Proof. We prove that (i) $\Rightarrow$ (ii). Assume that (i) holds and let $a,b>0$ be such that $aI\leq Y^*Y\leq bI$. Setting $Q:=Y^*Y$ and using the fact that $\Phi_{f,B}(I)\leq I$ and $a_\alpha\geq 0$, we have
$$\begin{aligned}\Phi_{f,A}(Q)&=\sum_{|\alpha|\geq 1} a_\alpha Y^*[P_\mathcal{G}(B_\alpha B_\alpha^*\otimes I_\mathcal{H})|_\mathcal{G}]Y\\ &=Y^*\left\{P_\mathcal{G}\left[\sum_{|\alpha|\geq 1} a_\alpha B_\alpha B_\alpha^*\otimes I\right]|_\mathcal{G}\right\}Y\\ &\leq Y^*Y=Q.\end{aligned}$$

Similar calculations reveal that
$$(id-\Phi_{f,A})^s(Q)=Y^*\left\{P_\mathcal{G}\left[(id-\Phi_{f,A})^s(I)\otimes I\right]|_\mathcal{G}\right\}Y\geq 0,\qquad s=1,\ldots,m.$$
Therefore, $Q\in C(f,A)^+$. Since $(B_1,\ldots,B_n)$ is a pure n-tuple in the noncommutative variety $\mathcal{V}^m_{f,\mathcal{P}}$, we have $\Phi^k_{f,B}(I)\to 0$ strongly, as $k\to\infty$. Taking into account that $\Phi^k_{f,A}(Q)=Y^*\left[P_\mathcal{G}(\Phi^k_{f,B}(I)\otimes I)|_\mathcal{G}\right]Y$ for $k\in\mathbb{N}$, we deduce that $\Phi^k_{f,A}(Q)\to 0$ strongly, as $k\to\infty$. Therefore item (ii) holds.

Now, we prove the implication (ii) $\Rightarrow$ (iii). Let $Q\in C(f,A)^+$ be an invertible operator such that $\Phi^k_{f,A}(Q)\to 0$ strongly, as $k\to\infty$. Set $R:=(id-\Phi_{f,A})^m(Q)$ and note that, using Lemma 1.3, we obtain
$$\begin{aligned}\sum_{k=0}^\infty \binom{k+m-1}{m-1}\Phi^k_{f,A}(R)&=Q-\text{SOT-}\lim_{k\to\infty}\sum_{j=0}^{m-1}\binom{k+j}{j}\Phi^{k+1}_{f,A}(id-\Phi_{f,A})^j(Q)\\ &=Q-\text{SOT-}\lim_{k\to\infty}\Phi^k_{f,A}(Q)=Q.\end{aligned}$$
Hence, we deduce item (iii). It remains to show that (iii) $\Rightarrow$ (i). Assume that item (iii) holds. Consider the noncommutative Berezin kernel $K^{(m)}_{f,A,R}:\mathcal{H}\to F^2(H_n)\otimes\mathcal{H}$ associated with with the quadruple (f,m,A,R) and defined by
$$K^{(m)}_{f,A,R}h=\sum_{\alpha\in\mathbb{F}_n^+}\sqrt{b_\alpha^{(m)}}e_\alpha\otimes R^{1/2}A_\alpha^*h,\qquad h\in\mathcal{H}.$$
According to Lemma 1.1 and using item (iii), we have
$$a\|h\|^2\leq\|K^{(m)}_{f,A,R}h\|^2=\sum_{k=0}^\infty\binom{k+m-1}{m-1}\langle\Phi^k_{f,A}(R)h,h\rangle\leq b\|h\|^2,\qquad h\in\mathcal{H}. \tag{3.1}$$

Consequently, the range of $K^{(m)}_{f,A,R}$ is a closed subspace of $F^2(H_n)\otimes\mathcal{H}$. On the other hand, we showed in Lemma 1.2 that

$$\operatorname{range} K^{(m)}_{f,A,R}\subseteq \mathcal{N}_{\mathcal{P}}\otimes\overline{R^{1/2}(\mathcal{H})}$$

and the noncommutative Berezin kernel $K_q:\mathcal{H}\to\mathcal{N}_{\mathcal{P}}\otimes\overline{R^{1/2}(\mathcal{H})}$ associated with the compatible tuple $q:=(f,m,A,R,\mathcal{P})$ and defined by

$$K_q:=\left(P_{\mathcal{N}_{\mathcal{P}}}\otimes I_{\overline{R^{1/2}(\mathcal{H})}}\right)K^{(m)}_{f,A,R},$$

has the property that

$$K_qA_i^*=(B_i^*\otimes I_{\overline{R^{1/2}(\mathcal{H})}})K_q,\qquad i=1,\ldots,n.$$

Consequently, the range of K_q is a closed subspace of $\mathcal{N}_{\mathcal{P}}\otimes\mathcal{H}$ and it is an invariant subspace under each operator $B_i^*\otimes I_{\mathcal{H}}$, $i=1,\ldots,n$. Since the operator $Y:\mathcal{H}\to\operatorname{range}K_q$ defined by $Yh:=K_qh$, $h\in\mathcal{H}$, is invertible, we have

$$A_i^*=Y^{-1}[(B_i^*\otimes I_{\overline{R^{1/2}(\mathcal{H})}})|_{\mathcal{G}}]Y,\qquad i=1,\ldots,n, \tag{3.2}$$

where $\mathcal{G}:=\operatorname{range}K_q$. This proves (i). The proof is complete. □

We remark that under the conditions of Theorem 3.2, part (iii), we can use relations (3.1) and (3.2) to show that the map $\Psi:\mathcal{A}_n(\mathcal{V}^m_{f,\mathcal{P}})\to B(\mathcal{H})$ defined by

$$\Psi(p(B_1,\ldots,B_n)):=p(A_1,\ldots,A_n)$$

is completely bounded with $\|\Psi\|_{cb}\leq\sqrt{\frac{b}{a}}$, and $\Phi^k_{f,A}(I)\to 0$ strongly, as $k\to\infty$. In the particular case when $m=1$ we have a converse of the latter result. Indeed, using Paulsen's similarity result [14] and the fact (which can be extracted from [28]) that any completely contractive representation of the noncommutative variety algebra $\mathcal{A}_n(\mathcal{V}^1_{f,\mathcal{P}})$ is generated by an n-tuple $(T_1,\ldots,T_n)\in\mathcal{V}^1_{f,\mathcal{P}}(\mathcal{H})$, we infer that $(A_1,\ldots,A_n)$ is simultaneously similar to an n-tuple $(T_1,\ldots,T_n)\in\mathcal{V}^1_{f,\mathcal{P}}(\mathcal{H})$ and $\Phi^k_{f,T}(I)\to\infty$ strongly, as $k\to\infty$. This proves our assertion.

Taking $R=I$ in Theorem 3.2, we can obtain the following analogue of Rota's model theorem, for similarity to n-tuples of operators in the noncommutative variety $\mathcal{V}^m_{f,\mathcal{P}}(\mathcal{H})$.

Corollary 3.3. *Let $\mathcal{P}$ be a set of noncommutative polynomials with $\mathcal{N}_{\mathcal{P}}\neq 0$ and let $A:=(A_1,\ldots,A_n)\in B(\mathcal{H})^n$ be such that $p(A_1,\ldots,A_n)=0$, $\quad p\in\mathcal{P}$, and*

$$\sum_{k=0}^{\infty}\binom{k+m-1}{m-1}\Phi^k_{f,A}(I)\leq bI$$

for some constant $b>0$. Then, there exists an invertible operator $Y:\mathcal{H}\to\mathcal{G}$ such that

$$A_i^*=Y^{-1}[(B_i^*\otimes I_{\mathcal{H}})|_{\mathcal{G}}]Y,\qquad i=1,\ldots,n,$$

where $\mathcal{G}\subseteq\mathcal{N}_{\mathcal{P}}\otimes\mathcal{H}$ is an invariant under each operator $B_i^*\otimes I_{\mathcal{H}}$ and $(B_1,\ldots,B_n)$ is the universal model associated with the noncommutative variety $\mathcal{V}^m_{f,\mathcal{P}}$.

Another consequence of Theorem 3.2 is the following analogue of Foiaş [9] and de Branges–Rovnyak [6] model theorem, for pure n-tuples of operators in $\mathcal{V}^m_{f,\mathcal{P}}(\mathcal{H})$.

Corollary 3.4. *An n-tuple of operators $T:=(T_1,\ldots,T_n)\in B(\mathcal{H})^n$ is in the noncommutative variety $\mathcal{V}^m_{f,\mathcal{P}}$ and it is pure, i.e., $\Phi^k_{f,T}(I)\to 0$ strongly, as $k\to\infty$, if and only if there exists a unitary operator $U:\mathcal{H}\to\mathcal{G}$ such that*

$$T_i^*=U^*[(B_i^*\otimes I_{\mathcal{D}})|_{\mathcal{G}}]U,\qquad i=1,\ldots,n,$$

where $\mathcal{D}:=\overline{[(id-\Phi_{f,T})^m(I)]^{1/2}(\mathcal{H})}$, the subspace $\mathcal{G}\subseteq\mathcal{N}_{\mathcal{P}}\otimes\mathcal{H}$ is invariant under each operator $B_i^*\otimes I_{\mathcal{D}}$ and $(B_1,\ldots,B_n)$ is the universal model associated with the noncommutative variety $\mathcal{V}^m_{f,\mathcal{P}}$.

Proof. Let $T := (T_1, \dots, T_n) \in \mathcal{V}^m_{f,\mathcal{P}}(\mathcal{H})$ be such that $\Phi^k_{f,T}(I) \to 0$ strongly, as $k \to \infty$. A closer look at the proof of Theorem 3.2, when $A = T$ and $Q = I_{\mathcal{H}}$, reveals that

$$K_q^* K_q = \sum_{k=0}^{\infty} \binom{k+m-1}{m-1} \Phi^k_{f,A}(R) = I,$$

where $R := (id - \Phi_{f,A})^m(I)$. Consequently, K_q is an isometry and the operator $U : \mathcal{H} \to K_q(\mathcal{H})$, defined by $Uh := K_q h$, $h \in \mathcal{H}$, is unitary. Now, one can use relation (3.2) to complete the proof. □

Next we obtain an analogue of Sz.-Nagy's similarity result [30].

Theorem 3.5. *Let $m \geq 1$, $f := \sum_{|\alpha|\geq 1} a_\alpha X_\alpha$ be a positive regular free holomorphic function and let $\mathcal{P}$ be a family of noncommutative polynomials. If $A := (A_1, \dots, A_n) \in B(\mathcal{H})^n$ is such that $\sum_{|\alpha|\geq 1} a_\alpha A_\alpha A_\alpha^*$ is convergent in the weak operator topology and $p(A_1, \dots, A_n) = 0, \quad p \in \mathcal{P}$, then the following statements are equivalent.*

(i) *There exist $(T_1, \dots, T_n) \in \mathcal{V}^m_{f,\mathcal{P}}(\mathcal{H})$ such that $(id - \Phi_{f,T})^m(I) = 0$ and an invertible operator $Y \in B(\mathcal{H})$ such that*
$$A_i = Y^{-1} T_i Y, \qquad i = 1, \dots, n.$$
(ii) *There exist positive constants $0 < c \leq d$ such that*
$$cI \leq \Phi^k_{f,A}(I) \leq dI, \qquad k \in \mathbb{N}.$$
(iii) *$\Phi_{f,A}$ is power bounded and there exists an invertible positive operator $Q \in B(\mathcal{H})$ such that $(id - \Phi_{f,A})^m(Q) = 0$.*

Proof. First we prove that (i) ⇔ (ii). Assume item (i) holds. Then we have

$$\begin{aligned}\Phi^k_{f,A}(I) &= Y^{-1} \Phi^k_{f,T}(YY^*) Y^{*-1} \\ &\leq \|YY^*\| Y^{-1} \Phi^k_{f,T}(I) Y^{*-1} \\ &\leq \|Y\|^2 \|Y^{-1}\|^2 I\end{aligned}$$

for any $k \in \mathbb{N}$. Now, we show that $\Phi_{f,T}(I) = I$. As in the proof of Theorem 3.2, we have

$$\sum_{p=0}^{q} \binom{p+m-1}{m-1} \Phi^p_{f,T}(id - \Phi_{f,T})^m(I) = I - \sum_{j=0}^{m-1} \binom{q+j}{j} \Phi^{q+1}_{f,T}(id - \Phi_{f,T})^j(I)$$

for any $q \in \mathbb{N}$. Consequently, if $(id - \Phi_{f,T})^m(I) = 0$, then

$$I = \lim_{q\to\infty} \sum_{j=0}^{m-1} \binom{q+j}{j} \Phi^{q+1}_{f,T}(id - \Phi_{f,T})^j(I).$$

Using Lemma 1.3, we deduce that $I = \lim_{q\to\infty} \Phi^q_{f,T}(I)$. Since $\Phi_{f,T}$ is a positive linear map and $\Phi_{f,T}(I) \leq I$, we have

$$I = \lim_{q\to\infty} \Phi^q_{f,T}(I) \leq \ldots \leq \Phi^2_{f,T}(I) \leq \Phi_{f,T}(I) \leq I.$$

Hence, we deduce that $\Phi_{f,T}(I) = I$. Consequently, we have

$$\begin{aligned}I = \Phi^k_{f,T}(I) &= Y \Phi^k_{f,A}(Y^{-1} Y^{*-1}) Y^* \\ &\leq \|Y^{-1} Y^{*-1}\| \, Y \Phi^k_{f,A}(I) Y^*\end{aligned}$$

for any $k \in \mathbb{N}$. Hence, we deduce that

$$Y^{-1} Y^{*-1} \leq \|Y^{-1}\|^2 \, \Phi^k_{f,A}(I), \qquad k \in \mathbb{N},$$

which implies

$$\Phi^k_{f,A}(I) \geq \frac{1}{\|Y^{-1}\|^2} Y^{-1} Y^{*-1} \geq \frac{1}{\|Y\|^2 \|Y^{-1}\|^2} I.$$

Therefore, we have proved that

$$\frac{1}{\|Y\|^2 \|Y^{-1}\|^2} I \leq \Phi^k_{f,A}(I) \leq \|Y\|^2 \|Y^{-1}\|^2 I, \qquad k \in \mathbb{N}.$$

Therefore, item (ii) holds. We prove now the implication (ii) $\Rightarrow$ (iii). Assume that item (ii) holds. For each $k\geq 1$, we define the operator

$$Q_k := \frac{1}{k}\sum_{j=0}^{k-1}\Phi_{f,A}^j(I)$$

and note that $cI\leq Q_k\leq dI$. Since the closed unit ball of $B(\mathcal{H})$ is weakly compact, there is a subsequence $\{Q_{k_j}\}_{j=1}^\infty$ weakly convergent to an operator $Q\in B(\mathcal{H})$. It is clear that Q is an invertible positive operator and $aI\leq Q\leq bI$. Since

$$Q_{k_j}-\Phi_{f,A}(Q_{k_j})=\frac{1}{k_j}I-\frac{1}{k_j}\Phi_{f,A}^{k_j}(I)$$

and taking into account that $\frac{1}{k_j}\Phi_{f,A}^{k_j}(I)\to 0$ in norm as $j\to\infty$, we get $\|Q_{k_j}-\Phi_{f,A}(Q_{k_j})\|\to 0$, as $j\to\infty$. On the other hand, according to Lemma 1.4, $\Phi_{f,A}$ is WOT-continuous on bounded sets. Now, using the fact that Q_{k_j} converges weakly to Q, we deduce that $\Phi_{f,A}(Q)=Q$, which implies $(id-\Phi_{f,A})^m(Q)=0$ and shows that item (iii) holds.

It remains to show that (iii) $\Rightarrow$ (i). Assume that $\Phi_{f,A}$ is power bounded and there exists an invertible positive operator $Q\in B(\mathcal{H})$ such that such that $(id-\Phi_{f,A})^m(Q)=0$. Since

$$\sum_{p=0}^{q}\binom{p+m-1}{m-1}\Phi_{f,A}^p(id-\Phi_{f,A})^m(Q)=Q-\sum_{j=0}^{m-1}\binom{q+j}{j}\Phi_{f,A}^{q+1}(id-\Phi_{f,A})^j(Q)$$

for any $q\in\mathbb{N}$, we deduce that

$$Q=\lim_{q\to\infty}\sum_{j=0}^{m-1}\binom{q+j}{j}\Phi_{f,A}^{q+1}(id-\Phi_{f,A})^j(Q).$$

Using Lemma 1.3, we deduce that $Q=\lim_{q\to\infty}\Phi_{f,A}^q(Q)$.

On the other hand, since $\Phi_{f,A}$ is a power bounded positive linear map with $(id-\Phi_{f,A})^m(Q)\geq 0$, we can use Lemma 1.4 to deduce that $(id-\Phi_{f,A})^s(Q)\geq 0$ for any $s=1,\ldots,m$. In paticular, we have $\Phi_{f,A}(Q)\leq Q$. Using the results above, we have

$$Q=\lim_{q\to\infty}\Phi_{f,A}^q(Q)\leq\ldots\leq\Phi_{f,A}^2(Q)\leq\Phi_{f,A}(Q)\leq Q.$$

Hence, we deduce that $\Phi_{f,A}(Q)=Q$. Set $T_i:=Q^{-1/2}A_iQ^{1/2}$ for $i=1,\ldots,n$ and note that

$$\begin{aligned}\sum_{|\alpha|\geq 1}a_\alpha T_\alpha T_\alpha^* &= Q^{-1/2}\left(\sum_{|\alpha|\geq 1}a_\alpha A_\alpha QA_\alpha^*\right)Q^{-1/2}\\ &= Q^{-1/2}QQ^{-1/2}=I,\end{aligned}$$

which implies item (i). The proof is complete. □

Now, we can obtain a noncommutative multivariable analogue of Douglas' similarity result [7].

Corollary 3.6. *If $A:=(A_1,\ldots,A_n)\in B(\mathcal{H})^n$ satisfies the conditions of Theorem 3.5 and*

$$\Phi_{f,A}^\infty(I):=\text{SOT-}\lim_{k\to\infty}\Phi_{f,A}^k(I)$$

exists, then the following statements are equivalent:

(i) *$\Phi_{f,A}^\infty(I)$ is invertible;*

(ii) *there exist $(T_1,\ldots,T_n)\in\mathcal{V}_{f,\mathcal{P}}^m(\mathcal{H})$ such that $(id-\Phi_{f,T})^m(I)=0$ and an invertible operator $Y\in B(\mathcal{H})$ such that*

$$A_i=Y^{-1}T_iY,\qquad i=1,\ldots,n.$$

In the particular case when $\Phi_{f,A}(I)\leq I$, the limit SOT-$\lim_{k\to\infty}\Phi_{f,A}^k(I)$ *always exists.*

Proof. Assume that item (i) holds. Since $\Phi_{f,A}$ is WOT-continuous on bounded sets (see Lemma 1.4) and the limit SOT-$\lim_{k\to\infty} \Phi^k_{f,A}(I)$ exists, we have $\Phi_{f,A}(\Phi^\infty_{f,A}(I)) = \Phi^\infty_{f,A}(I)$. Taking into account that $\Phi^\infty_{f,A}(I)$ is invertible, item (ii) follows from Theorem 3.5. Conversely, assume that item (ii) holds. Then Theorem 3.5, implies $cI \le \Phi^k_{f,A}(I) \le dI$ for any $k \in \mathbb{N}$. Hence, the operator $\Phi^\infty_{f,A}(I)$ is invertible, and the proof is complete. $\square$

Given $A, B \in B(\mathcal{H})$ two self-adjoint operators, we say that $A < B$ if $B - A$ is positive and invertible, i.e., there exists a constant $\gamma > 0$ such that $\langle (B-A)h, h\rangle \ge \gamma \|h\|^2$ for any $h \in \mathcal{H}$. Note that $C \in B(\mathcal{H})$ is a strict contraction ($\|C\| < 1$) if and only if $C^*C < I$.

A version of Rota's model theorem (see [29], [10]) asserts that any operator with spectral radius less than one is similar to a strict contraction. In what follows we present an analogue of this result in our multivariable noncommutative setting.

Theorem 3.7. *Let $m \ge 1$, $f := \sum_{|\alpha|\ge 1} a_\alpha X_\alpha$ be a positive regular free holomorphic function and let $\mathcal{P}$ be a family of noncommutative polynomials. If $A := (A_1, \ldots, A_n) \in B(\mathcal{H})^n$ is such that $\sum_{|\alpha|\ge 1} a_\alpha A_\alpha A^*_\alpha$ is convergent in the weak operator topology and $p(A_1, \ldots, A_n) = 0, \quad p \in \mathcal{P}$, then the following statements are equivalent.*

(i) *There exist $T := (T_1, \ldots, T_n) \in \mathcal{V}^m_{f,\mathcal{P}}(\mathcal{H})$ with $(id - \Phi_{f,T})^m(I) > 0$ and an invertible operator $Y \in B(\mathcal{H})$ such that*
$$A_i = Y^{-1} T_i Y, \qquad i = 1, \ldots, n.$$
(ii) *$\Phi_{f,A}$ is power bounded and there exists a positive operator $Q \in B(\mathcal{H})$ such that*
$$(id - \Phi_{f,A})^m(Q) > 0.$$
(iii) $r_f(A_1, \ldots, A_n) < 1$.
(iv) $\lim_{k\to\infty} \|\Phi^k_{f,A}(I)\| = 0$.
(v) *$\Phi_{f,A}$ is power bounded and there is an invertible positive operator $R \in B(\mathcal{H})$, such that the equation*
$$(id - \Phi_{f,A})^m(X) = R$$
has a positive solution X in $B(\mathcal{H})$.

Moreover, in this case, for any invertible positive operator $R \in B(\mathcal{H})$, the equation $(id - \Phi_{f,A})^m(X) = R$ has a unique positive solution, namely,
$$X := \sum_{k=0}^\infty \binom{k+m-1}{m-1} \Phi^k_{f,A}(R),$$
where the convergence is in the uniform topology, which is an invertible operator.

Proof. First we prove the equivalence (i) $\Leftrightarrow$ (ii). Assume that (i) holds and $(id - \Phi_{f,T})^m(I) \ge cI$ for some $c > 0$. Then we have
$$Y\left[(id - \Phi_{f,A})^m(Y^{-1}(Y^{-1})^*)\right] Y^* \ge cI.$$
Setting $Q := Y^{-1}(Y^{-1})^*$ we deduce that $(id - \Phi_{f,A})^m(Q) > 0$. The fact that $\Phi_{f,A}$ is power bounded is due to Proposition 3.1.

Conversely, assume that item (ii) holds and let $Q \in B(\mathcal{H})$ be a positive operator such that $(id - \Phi_{f,A})^m(Q) > 0$. Since $\Phi_{f,A}$ is power bounded, Lemma 1.3 implies $(id - \Phi_{f,A})^s(Q) \ge 0$, $s = 1, \ldots, m$. On the other hand, since $\Phi_{f,A}$ is a positive linear map, we deduce that
$$0 < (id - \Phi_{f,A})^m(Q) \le \cdots \le (id - \Phi_{f,A})(Q) \le Q.$$
Therefore, Q is an invertible positive operator. Since
$$(id - \Phi_{f,A})^m(Q) \ge bI$$
for some constant $b > 0$, we can choose $c > 0$ such that $bI \ge cQ$, and deduce that
$$Q^{-1/2}[(id - \Phi_{f,A})^m(Q)]Q^{-1/2} \ge cI.$$
Setting $T_i := Q^{-1/2} A_i Q^{1/2}$, $i = 1, \ldots, n$, the latter inequality implies $(id - \Phi_{f,T})^m(I) > 0$. Since $\Phi_{f,A}$ is power bounded, so is $\Phi_{f,T}$. As above, using again Lemma 1.3 we obtain $(id - \Phi_{f,T})^s(I) > 0$,

$s = 1, 2, \ldots, m$, which shows that $T \in \mathbf{D}_f^m(\mathcal{H})$. Since $p(A_1, \ldots, A_n) = 0$, $p \in \mathcal{P}$, we deduce that $T \in \mathcal{V}_{f,\mathcal{P}}^m(\mathcal{H})$. Therefore, item (i) holds.

Now we prove the equivalence (iii) $\Leftrightarrow$ (iv). Assume that item (iii) holds and let $a > 0$ be such that $r(A_1, \ldots, A_n) < a < 1$. Then there is $m_0 \in \mathbb{N}$ such that $\|\Phi_{f,A}^k(I)\| \leq a^k$ for any $k \geq m_0$. This clearly implies condition (iv). Now, we assume that (iv) holds. Note that

$$\begin{aligned} r_f(A_1, \ldots, A_n)^j &= \lim_{k\to\infty} \left[\|\Phi_{f,A}^{jk}(I)\|^{1/2kj}\right]^j \\ &= \lim_{k\to\infty} \|\Phi_{f,A}^{j(k-1)}(\Phi_{f,A}^j(I))\|^{1/2k} \\ &\leq \lim_{k\to\infty} \left(\|\Phi_{f,A}^j(I)\|^k\right)^{1/2k} = \|\Phi_{f,A}^j(I)\|^{1/2} \end{aligned}$$

for any $j \in \mathbb{N}$. Consequently, $r_f(A_1, \ldots, A_n) < 1$, so item (iii) holds. The implication (v) $\Rightarrow$ (ii) is obvious. In what follows we prove that (i) $\Rightarrow$ (iii). Assume that there exists $T := (T_1, \ldots, T_n) \in \mathcal{V}_{f,\mathcal{P}}^m(\mathcal{H})$ with $(id - \Phi_{f,T})^m(I) > 0$ and an invertible operator $Y \in B(\mathcal{H})$ such that

$$A_i = Y^{-1} T_i Y, \qquad i = 1, \ldots, n.$$

Recall that under these conditions we have, in particular, $\|\Phi_{f,A}(I)\| < 1$. On the other hand, note that

$$\begin{aligned} r_f(T_1, \ldots, T_n) &= r_f(YA_1Y^{-1}, \ldots, YA_nY^{-1}) \\ &= \lim_{k\to\infty} \|\Phi_{f,YAY^{-1}}^k(I)\|^{1/2k} \\ &\leq \lim_{k\to\infty} \|Y\|^{1/k} \|\Phi_{f,A}^k(I)\|^{1/2k} \\ &= r_f(A_1, \ldots, A_n). \end{aligned}$$

Hence, applying this inequality when Y is replaced by its inverse, we deduce that

$$\begin{aligned} r_f(A_1, \ldots, A_n) &= r_f\left(Y^{-1}(YA_1Y^{-1})Y, \ldots, Y^{-1}(YA_nY^{-1})Y\right) \\ &\leq r_f(YA_1Y^{-1}, \ldots, YA_nY^{-1}) = r_f(T_1, \ldots, T_n). \end{aligned}$$

Therefore, we have

$$\begin{aligned} r_f(A_1, \ldots, A_n) &= r_f(T_1, \ldots, T_n) \\ &= \lim_{k\to\infty} \|\Phi_{f,T}^k(I)\|^{1/2k} \leq \|\Phi_{f,T}(I)\|^{1/2} < 1, \end{aligned}$$

which shows that item (iii) holds. Now, we prove the implication (iii) $\Rightarrow$ (v). To this end, assume that $r_f(A_1, \ldots, A_n) < 1$ and let $R \in B(\mathcal{H})$ be an invertible positive operator. We have

$$\frac{1}{\|R^{-1}\|} I \leq R \leq \sum_{k=0}^\infty \binom{k+m-1}{m-1} \Phi_{f,A}^k(R) \leq \left(\|R\| \sum_{k=0}^\infty \binom{k+m-1}{m-1} \|\Phi_{f,A}^k(I)\|\right) I.$$

Note that

$$\lim_{k\to\infty} \left[\binom{k+m-1}{m-1} \|\Phi_{f,A}^k(I)\|\right]^{1/2k} = r_f(T_1, \ldots, T_n) < 1.$$

Consequently,

$$aI \leq \sum_{k=0}^\infty \binom{k+m-1}{m-1} \Phi_{f,A}^k(R) \leq bI \tag{3.3}$$

for some constants $0 < a < b$, where the convergence of the series is in the operator norm topology. Now, we can prove that

$$(id - \Phi_{f,A})^m \left[\sum_{k=0}^\infty \binom{k+m-1}{m-1} \Phi_{f,A}^k(R)\right] = R.$$

Indeed, since $(id - \Phi_{f,A})\Phi_{f,A} = \Phi_{f,A}(id - \Phi_{f,A})$, we can use Lemma 1.3 and the fact that

$$0 \leq \lim_{k\to\infty} \|\Phi_{f,A}^k(R)\| \leq \|R\| \lim_{k\to\infty} \|\Phi_{f,A}^k(I)\| = 0$$

to obtain

$$
\begin{aligned}
(id-\Phi_{f,A})^m\left[\sum_{k=0}^\infty \binom{k+m-1}{m-1}\Phi_{f,A}^k(R)\right] &= \sum_{k=0}^\infty \binom{k+m-1}{m-1}\Phi_{f,A}^k(id-\Phi_{f,A})^m(R)\\
&= R-\text{SOT-}\lim_{k\to\infty}\sum_{i=0}^{m-1}\binom{k+i}{i}\Phi_{f,A}^{k+1}(id-\Phi_{f,A})^i(R)\\
&= R-\text{SOT-}\lim_{k\to\infty}\Phi_{f,A}^k(R)=R
\end{aligned}
$$

Consequently, and due to relation (3.3),

$$X:=\sum_{k=0}^\infty \binom{k+m-1}{m-1}\Phi_{f,A}^k(R)$$

is an invertible positive solution of the equation $(id-\Phi_{f,A})^m(X)=R$. Therefore item (v) holds.

To prove the last part of the theorem, let $X'\geq 0$ be an invertible operator such $(id-\Phi_{f,A})^m(X')=R$, where $R\geq 0$ is a fixed arbitrary invertible operator. Then, as above, we have

$$
\begin{aligned}
\sum_{k=0}^\infty \binom{k+m-1}{m-1}\Phi_{f,A}^k(R) &= \sum_{k=0}^\infty \binom{k+m-1}{m-1}\Phi_{f,A}^k(id-\Phi_{f,A})^m(X')\\
&= X'-\text{SOT-}\lim_{k\to\infty}\Phi_{f,A}^k(X')=X'.
\end{aligned}
$$

Here we used that $\|\Phi_{f,A}^k(X')\|\leq \|X'\|\|\Phi_{f,A}^k(I)\|\to 0$, as $k\to\infty$. Therefore there is unique positive solution of the inequality $(id-\Phi_{f,A})^m(X)=R$. The proof is complete. □

Now we can obtain the following multivariable generalization of Rota's similarity result (see Paulsen's book [15]).

Corollary 3.8. *Under the hypotheses of Theorem 3.7, if the joint spectral radius* $r_f(A_1,\ldots,A_n)<1$, *then the* n*-tuple*

$$T:=(P^{-1/2}A_1P^{1/2},\ldots,P^{-1/2}A_nP^{1/2})$$

is in the noncommutative variety $\mathcal{V}^m_{f,\mathcal{P}}(\mathcal{H})$ *and* $(id-\Phi_{f,T})^m(I)>0$, *where*

$$P:=\sum_{k=0}^\infty \binom{k+m-1}{m-1}\Phi_{f,A}^k(I)$$

is convergent in the operator norm topology and

$$\|P^{1/2}\|\|P^{-1/2}\|\leq\left(\sum_{k=0}^\infty \binom{k+m-1}{m-1}\|\Phi_{f,A}^k(I)\|\right)^{1/2}.$$

In particular, if f *is a positive regular noncommutative polynomial, then* P *is in the* C^**-algebra generated by* $A_1,\ldots,A_n$ *and the identity.*

Proof. Since

$$\lim_{k\to\infty}\left[\binom{k+m-1}{m-1}\|\Phi_{f,A}^k(I)\|\right]^{1/2k}=r_f(T_1,\ldots,T_n)<1,$$

the series $\sum_{k=0}^\infty \binom{k+m-1}{m-1}\|\Phi_{f,A}^k(I)\|$ is convergent and we have

$$I\leq P\leq \sum_{k=0}^\infty \binom{k+m-1}{m-1}\|\Phi_{f,A}^k(I)\|,$$

which implies the upper bound estimation for $\|P^{1/2}\|\|P^{-1/2}\|$. A closer look at the proof of Theorem 3.7 and taking $R=I$ leads to the desired result. The last part of this corollary is now obvious. □

Using Theorem 3.7 and Theorem 3.2, we deduce the following result.

Corollary 3.9. *Let $A := (A_1, \ldots, A_n) \in B(\mathcal{H})^n$ be under the hypotheses of Theorem 3.7. Then the following statements hold.*

(i) *If $r_f(A_1, \ldots, A_n) = 0$, then, for any $\epsilon > 0$, $(A_1, \ldots, A_n)$ is jointly similar to an n-tuple of operators $(T_1, \ldots, T_n) \in \epsilon \mathcal{V}^m_{f,\mathcal{P}}(\mathcal{H})$.*

(ii) *If there exist positive constants $0 < a \leq b$ and a positive operator $R \in B(\mathcal{H})$ such that*

$$aI \leq \sum_{k=0}^{\infty} \binom{k+m-1}{m-1} \Phi^k_{f,A}(R) \leq bI,$$

then $(A_1, \ldots, A_n)$ is jointly similar to an n-tuple of operators $(T_1, \ldots, T_n) \in \mathcal{V}^m_{f,\mathcal{P}}(\mathcal{H})$. If, in addition, R is invertible, then $(id - \Phi_{f,T})^m(I) > 0$.

The next result provides necessary and sufficient conditions for an n-tuple of operators be similar to an n-tuple in the noncommutative variety $\mathcal{V}^m_{f,\mathcal{P}}$, $m \geq 1$.

Theorem 3.10. *Let $m \geq 1$, $f := \sum_{|\alpha| \geq 1} a_\alpha X_\alpha$ be a positive regular free holomorphic function and let $\mathcal{P}$ be a family of noncommutative polynomials. If $A := (A_1, \ldots, A_n) \in B(\mathcal{H})^n$ is such that $\sum_{|\alpha| \geq 1} a_\alpha A_\alpha A^*_\alpha$ is convergent in the weak operator topology and $p(A_1, \ldots, A_n) = 0, \quad p \in \mathcal{P}$, then the following statements are equivalent.*

(i) *There exist an n-tuple $(T_1, \ldots, T_n) \in \mathcal{V}^m_{f,\mathcal{P}}(\mathcal{H})$ and an invertible operator $Y \in B(\mathcal{H})$ such that*

$$A_i = Y^{-1} T_i Y, \qquad i = 1, \ldots, n.$$

(ii) *$\Phi_{f,A}$ is power bounded and there is an invertible positive operator $R \in B(\mathcal{H})$ such that*

$$(id - \Phi_{f,A})^m(R) \geq 0.$$

If, in addition, $m = 1$ and $\mathcal{P}$ is a set of homogeneous polynomials, then the statements above are equivalent to the following:

(iii) *the map $\Psi : \mathcal{A}_n(\mathcal{V}^1_{f,\mathcal{P}}) \to B(\mathcal{H})$ defined by*

$$\Psi(p(B_1, \ldots, B_n)) := p(A_1, \ldots, A_n)$$

is completely bounded, where $\mathcal{A}_n(\mathcal{V}^1_{f,\mathcal{P}})$ is the noncommutative variety algebra.

Proof. The proof of the equivalence (i) $\Leftrightarrow$ (ii) is similar to the proof of the same equivalence from Theorem 3.7. Consider the case $m = 1$. If item (i) holds, then

$$p(A_1, \ldots, A_n) = Y p(T_1, \ldots, T_n) Y^{-1}$$

for any noncommutative polynomial p. Using the noncommutative von Neumann inequality for $\mathcal{V}^1_{f,\mathcal{P}}(\mathcal{H})$, we deduce $\|\Psi\|_{cb} \leq \|Y\| \|Y^{-1}\|$. Now, if we assume that item (iii) holds, then using Paulsen's similarity result [14] and the fact (see [28]) that any completely contractive representation of the noncommutative variety algebra $\mathcal{A}_n(\mathcal{V}^1_{f,\mathcal{P}})$ is generated by an n-tuple $(T_1, \ldots, T_n) \in \mathcal{V}^1_{f,\mathcal{P}}(\mathcal{H})$, we infer that $(A_1, \ldots, A_n)$ is simultaneously similar to an n-tuple $(T_1, \ldots, T_n) \in \mathcal{V}^1_{f,\mathcal{P}}(\mathcal{H})$. The proof is complete. $\square$

4. Joint invariant subspaces and triangulations for n-tuples of operators in noncommutative varieties

In this section, we obtain Wold type decompositions and prove the existence of triangulations of type

$$\begin{pmatrix} C_{\cdot 0} & 0 \\ * & C_{\cdot 1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} C_c & 0 \\ * & C_{cnc} \end{pmatrix}$$

for any n-tuple of operators in the noncommutative variety $\mathcal{V}^1_{f,\mathcal{P}}(\mathcal{H})$. As consequences, we show that certain classes of n-tuples of operators in $\mathcal{V}^1_{f,\mathcal{P}}(\mathcal{H})$ have non-trivial joint invariant subspaces.

Theorem 4.1. *Let $f := \sum_{|\alpha|\geq 1} a_\alpha X_\alpha$ be a positive regular free holomorphic function and $m \geq 1$. Let $A := (A_1, \ldots, A_n) \in B(\mathcal{H})^n$ be such that $\sum_{|\alpha|\geq 1} a_\alpha A_\alpha A_\alpha^*$ is convergent in the weak operator topology and $\Phi_{f,A}$ is power bounded. If $D \in B(\mathcal{H})$ be a positive operator such that $(id - \Phi_{f,A})^m(D) \geq 0$. Then the subspaces*

$$\ker D, \quad \{h \in \mathcal{H} : \lim_{k\to\infty} \Phi^k_{f,A}(D)h = 0\}, \quad \text{and} \quad \{h \in \mathcal{H} : \Phi^k_{f,A}(D)h = Dh \text{ for all } k \in \mathbb{N}\}$$

are invariant under each operator A_i^*, $i = 1, \ldots, n$.

In particular, if $\mathcal{M}$ is a subspace of $\mathcal{H}$ and $(id - \Phi_{f,A})^m(P_\mathcal{M}) \geq 0$, where $P_\mathcal{M}$ is the orthogonal projection onto $\mathcal{M}$, then $\mathcal{M}$ is invariant under each operator A_i.

Proof. Due to Lemma 1.3, we have $\Phi_{f,A}(D) \leq D$. Consequently, for any $h \in \ker D$,

$$0 \leq \sum_{|\alpha|\geq 1} \langle a_\alpha A_\alpha D A_\alpha^* h, h\rangle \leq \langle Dh, h\rangle = 0.$$

Hence, $\|a_{g_i} D^{1/2} A_i^* h\| = 0$ for $i = 1, \ldots, n$. Since $a_{g_i} \neq 0$, we deduce that $A_i^* h \in \ker D$. Therefore, $\ker D$ is invariant under each operator A_i^*. Now, let $D = B + C$ be the canonical decomposition of D with respect to $\Phi_{f,A}$. According to Theorem 2.1, we have

$$C \geq 0, \quad (id - \Phi_{f,A})^m(C) \geq 0, \quad \text{SOT-}\lim_{k\to\infty} \Phi^k_{f,A}(C) = 0,$$

and

$$B = \text{SOT-}\lim_{k\to\infty} \Phi^k_{f,A}(D), \quad \Phi_{f,A}(B) = B.$$

Now, due to the first part of this theorem, applied to B and C, respectively, the subspaces $\ker B$ and $\ker C$ are invariant under each each operator A_i^*, $i = 1, \ldots, n$. Note that

$$\ker B = \{h \in \mathcal{H} : \lim_{k\to\infty} \Phi^k_{f,A}(D)h = 0\}.$$

Since $\Phi_{f,A}(D) \leq D$, it is easy to see that

$$\ker C = \{h \in \mathcal{H} : \lim_{k\to\infty} \Phi^k_{f,A}(D)h = Dh\} = \{h \in \mathcal{H} : \Phi^k_{f,A}(D)h = Dh,\ k \in \mathbb{N}\}.$$

Taking $D := P_\mathcal{M}$, we obtain the last part of the theorem. The proof is complete. □

An interesting consequence of Proposition 4.1 is the following.

Corollary 4.2. *Let $T := (T_1, \ldots, T_n) \in \mathbf{D}^m_f(\mathcal{H})$ be such that $(id - \Phi_{f,T})^m(I) = I$ and let $\mathcal{M} \subseteq \mathcal{H}$ be a subspace. Then the following statements hold.*

(i) *$\mathcal{M}$ is an invariant subspace under each operator T_i, $i = 1, \ldots, n$, if and only if $\Phi_{f,T}(P_\mathcal{M}) \leq P_\mathcal{M}$.*
(ii) *$\mathcal{M}$ is reducing under each operator T_i, $i = 1, \ldots, n$, if and only if $\Phi_{f,T}(P_\mathcal{M}) = P_\mathcal{M}$.*

Proof. Due to Theorem 4.1, if $\Phi_{f,T}(P_\mathcal{M}) \leq P_\mathcal{M}$, then the subspace $\mathcal{M}$ is invariant under under each operator T_i, $i = 1, \ldots, n$. Conversely, assume $\mathcal{M}$ is invariant under under each T_i, $i = 1, \ldots, n$. Then $P^\perp_\mathcal{M} T_i P^\perp_\mathcal{M} = P^\perp_\mathcal{M} T_i$, where $P^\perp_\mathcal{M} := I - P_\mathcal{M}$. As seen before in this paper, the condition $(id - \Phi_{f,T})^m(I) = 1$ implies $\Phi_{f,T}(I) = I$. Consequently, we have

$$\Phi_{f,T}(P^\perp_\mathcal{M})P^\perp_\mathcal{M} = \Phi_{f,T}(I)P^\perp_\mathcal{M} = P^\perp_\mathcal{M} = P^\perp_\mathcal{M}\Phi_{f,T}(I)P^\perp_\mathcal{M} = P^\perp_\mathcal{M}\Phi_{f,T}(P^\perp_\mathcal{M})P^\perp_\mathcal{M}.$$

Since the operators $\Phi_{f,T}(P^\perp_\mathcal{M})$ and $I - P^\perp_\mathcal{M}$ are positive and commuting, we have

$$\Phi_{f,T}(P^\perp_\mathcal{M}) - P^\perp_\mathcal{M} = \Phi_{f,T}(P^\perp_\mathcal{M})(I - P^\perp_\mathcal{M}) \geq 0.$$

Consequently, $\Phi_{f,T}(P^\perp_\mathcal{M}) \geq P^\perp_\mathcal{M}$. Since $\Phi_{f,T}(I) = I$, we infer that $\Phi_{f,T}(P_\mathcal{M}) \leq P_\mathcal{M}$. Therefore, item (i) holds. To prove (ii), note that due to part (i), $\mathcal{M}$ is reducing under each operator T_i, $i = 1, \ldots, n$, if and only if $\Phi_{f,T}(P_\mathcal{M}) \leq P_\mathcal{M}$ and $\Phi_{f,T}(P^\perp_\mathcal{M}) \leq P^\perp_\mathcal{M}$. Since $\Phi_{f,T}(I) = I$, the result follows. □

Let f be a positive regular free holomorphic function, $m \geq 1$, and let $K^{(m)}_{f,T,R}$ be the noncommutative Berezin kernel associated with the noncommutative domain $\mathbf{D}^m_f$, i.e., associated with the quadruple

$q := (f, m, T, R)$, where $R := (id - \Phi_{f,T})^m(I)$. We remark that, as in the proof of Theorem 1.5, one can use Lemma 1.1 and Lemma 1.3 to obtain relation

$$\left(K_{f,T,R}^{(m)}\right)^* K_{f,T,R}^{(m)} = \sum_{k=0}^{\infty} \binom{k+m-1}{m-1} \Phi_{f,T}^k(R) = I - Q_{f,T},$$

where $Q_{f,T} := \text{SOT-}\lim_{k\to\infty} \Phi_{f,T}^k(I)$.

Lemma 4.3. *Let $f := \sum_{|\alpha|\geq 1} a_\alpha X_\alpha$ be a positive regular free holomorphic function and $m \geq 1$. If $(T_1, \ldots, T_n)$ is an n-tuple of operators in the noncommutative domain $\mathbf{D}_f^m(\mathcal{H})$, then the limit*

$$Q_{f,T} := \text{SOT-}\lim_{k\to\infty} \Phi_{f,T}^k(I)$$

exists and we have

$$\begin{aligned} \ker Q_{f,T} &= \{h \in \mathcal{H} : \lim_{k\to\infty} \left\langle \Phi_{f,T}^k(I)h, h\right\rangle = 0\} \\ &= \{h \in \mathcal{H} : \|K_{f,T,R}^{(m)} h\| = \|h\|\} \\ &= \ker \left[I - \left(K_{f,T,R}^{(m)}\right)^* K_{f,T,R}^{(m)}\right] \end{aligned}$$

and

$$\begin{aligned} \ker(I - Q_{f,T}) &= \{h \in \mathcal{H} : \Phi_{f,T}^k(I)h = h, \ k \in \mathbb{N}\} \\ &= \{h \in \mathcal{H} : \left\langle \Phi_{f,T}^k(I)h, h\right\rangle = \|h\|^2, \ k \in \mathbb{N}\} \\ &= \ker K_{f,T,R}^{(m)}, \end{aligned}$$

where $K_{f,T,R}^{(m)}$ in the noncommutative Berezin kernel associated with the noncommutative domain $\mathbf{D}_f^m$.

Proof. Since $\Phi_{f,T}(I) \leq I$, the sequence of positive operators $\Phi_{f,T}^k(I)$ is decreasing. Consequently, the operator $Q_{f,T}$ exists and has the properties: $0 \leq Q_{f,T} \leq I$ and $\Phi_{f,T}(Q_{f,T}) = Q_{f,T}$. Using relation $\left(K_{f,T,R}^{(m)}\right)^* K_{f,T,R}^{(m)} = I - Q_{f,T}$ we deduce some of the equalities above. The others are fairly easy. □

Now we can obtain the following Wold type decomposition for n-tuples of operators in the noncommutative domain $\mathbf{D}_f^m(\mathcal{H})$.

Theorem 4.4. *Let $f := \sum_{|\alpha|\geq 1} a_\alpha X_\alpha$ be a positive regular free holomorphic function and $m \geq 1$. If $(T_1, \ldots, T_n)$ is an n-tuple of operators in the noncommutative domain $\mathbf{D}_f^m(\mathcal{H})$ and*

$$Q_{f,T} := \text{SOT-}\lim_{k\to\infty} \Phi_{f,T}^k(I),$$

then the space $\mathcal{H}$ admits a decomposition of the form

$$\mathcal{H} = \mathcal{M} \oplus \ker Q_{f,T} \oplus \ker(I - Q_{f,T}),$$

where the subspaces $\ker Q_{f,T}$ and $\ker(I - Q_{f,T})$ are invariant under each operator T_i^*, $i = 1, \ldots, n$.

Proof. According to Lemma 4.3, the operator $Q_{f,T}$ exists and has the properties: $0 \leq Q_{f,T} \leq I$ and $\Phi_{f,T}(Q_{f,T}) = Q_{f,T}$. Since

$$\mathcal{H} = \overline{Q_{f,T}(\mathcal{H})} \oplus \ker Q_{f,T} \quad \text{and} \quad \ker(I - Q_{f,T}) \subseteq Q_{f,T}(\mathcal{H}),$$

we obtain the desired decomposition. The fact that $\ker Q_{f,T}$ is an invariant subspace under each operator T_i^*, $i = 1, \ldots, n$, follows from Theorem 4.1. Now we assume that $Q_{f,T} \neq 0$. According to Lemma 1.1,

$$K_{f,T,R}^{(m)} T_i^* = (W_i^* \otimes I_{\overline{R^{1/2}(\mathcal{H})}}) K_{f,T,R}^{(m)}, \qquad i = 1, \ldots, n.$$

Hence $\ker K_{f,T,R}^{(m)}$ is an invariant subspace under each operator T_i^*, $i = 1, \ldots, n$. On the other hand, due to Lemma 4.3, we have $\ker K_{f,T,R}^{(m)} = \ker(I - Q_{f,T})$. The proof is complete. □

We have another proof of the fact that $\ker(I-Q_{f,T})$ are invariant under each operator T_i^*, $i=1,\dots,n$, which does not use the noncommutative Berezin kernel. Indeed, assume that $Q_{f,T}\neq 0$. Then

$$\langle Q_{f,T}h,h\rangle = \langle \Phi_{f,T}^k(Q_{f,T})h,h\rangle \leq \|Q_{f,T}\|\langle \Phi_{f,T}^k(I)h,h\rangle, \qquad h\in\mathcal{H}, k\in\mathbb{N}.$$

Taking the limit as $k\to\infty$, we obtain

$$\langle Q_{f,T}h,h\rangle \leq \|Q_{f,T}\|^2\left\langle h,h\right\rangle.$$

Hence, $\|Q_{f,T}^{1/2}\|\leq \|Q_{f,T}^{1/2}\|^2=\|Q_{f,T}\|\leq 1$ and, consequently, we deduce that $\|Q_{f,T}^{1/2}\|=0$ or $\|Q_{f,T}^{1/2}\|=1$. Since $Q_{f,T}\neq 0$, we must have $\|Q_{f,T}\|=1$. We show now that the set $\ker(I-Q_{f,T})$ is invariant under each T_i^*, $i=1,\dots,n$. Indeed, note that $I-Q_{f,T}\geq 0$ and

$$\Phi_{f,T}(I-Q_{f,T})=\Phi_{f,T}(I)-\Phi_{f,T}(Q_{f,T})\leq I-Q_{f,T}.$$

Applying Theorem 4.1 to the positive operator $I-Q_{f,T}$, the result follows.

Let $m\geq 1$, $f:=\sum_{|\alpha|\geq 1}a_\alpha X_\alpha$ be a positive regular free holomorphic function and let $\mathcal{P}$ be a family of noncommutative polynomials with $\mathcal{N}_\mathcal{P}\neq 0$. Let $(T_1,\dots,T_n)$ be an n-tuple of operators in the noncommutative variety $\mathcal{V}_{f,\mathcal{P}}^m(\mathcal{H})$ and let K_q be the Berezin kernel associated with $\mathcal{V}_{f,\mathcal{P}}^m(\mathcal{H})$, i.e., associated with the tuple $q=(f,m,T,R,\mathcal{P})$, where $R:=(id-\Phi_{f,T})^m(I)$. Under these conditions, Lemma 1.2 and Lemma 1.3 and imply $K_q^*K_q=I-Q_{f,T}$. Consequently, one can obtain the following version of Theorem 4.4.

Corollary 4.5. *The space $\mathcal{H}$ admits an orthogonal decomposition*

$$\mathcal{H}=\mathcal{M}\oplus\ker(I-K_q^*K_q)\oplus\ker K_q,$$

where the subspaces $\ker(I-K_q^*K_q)$ and $\ker K_q$ are invariant under each operator T_i^*, $i=1,\dots,n$.

An interesting consequence of Theorem 4.4 is the following Wold type decomposition.

Corollary 4.6. *Let $m\geq 1$, $f:=\sum_{|\alpha|\geq 1}a_\alpha X_\alpha$ be a positive regular free holomorphic function and let $\mathcal{P}$ be a family of noncommutative polynomials with $\mathcal{N}_\mathcal{P}\neq 0$. Let $(T_1,\dots,T_n)$ be an n-tuple of operators in the noncommutative variety $\mathcal{V}_{f,\mathcal{P}}^m(\mathcal{H})$ and let K_q be the Berezin kernel associated with $\mathcal{V}_{f,\mathcal{P}}^m(\mathcal{H})$. Then the following statements are equivalent:*

(i) *the Hilbert space $\mathcal{H}$ admits the orthogonal decompositions*

$$\mathcal{H}=\ker Q_{f,T}\oplus\ker(I-Q_{f,T})=\ker(I-K_q^*K_q)\oplus\ker K_q;$$

(ii) *$Q_{f,T}$ is an orthogonal projection;*
(iii) *the noncommutative Berezin kernel K_q is a partial isometry.*

In this case, the subspaces

$$\ker Q_{f,T}=\ker(I-K_q^*K_q) \quad \text{and} \quad \ker(I-Q_{f,T})=\ker K_q$$

are reducing for each operator T_i, $i=1,\dots,n$.

Proof. Since $Q_{f,T}$ is a positive operator, it is well-known that

$$\ker[Q_{f,T}-Q_{f,T}^2]=\ker Q_{f,T}\oplus\ker(I-Q_{f,T}).$$

On the other hand, note that $\ker[Q_{f,T}-Q_{f,T}^2]=\mathcal{H}$ if and only if $Q_{f,T}$ is an orthogonal projection. Using the results preceding this corollary, we can complete the proof. □

Let $m=1$, $p=\sum_{|\alpha|\geq 1}a_\alpha X_\alpha$ be a positive regular noncommutative polynomial and let $\mathcal{P}$ be a set of noncommutative polynomials such that $1\in\mathcal{N}_\mathcal{P}$. In [28], using standard theory of representations of C^*-algebras, we obtained the following Wold type decomposition for non-degenerate $*$-representations of the unital C^*-algebra $C^*(B_1,\dots,B_n)$, generated by the constrained weighted shifts associated with the noncommutative variety $\mathcal{V}_{p,\mathcal{P}}^1$, and the identity. If $\pi: C^*(B_1,\dots,B_n)\to B(\mathcal{K})$ is a non-degenerate $*$-representation of $C^*(B_1,\dots,B_n)$ on a separable Hilbert space $\mathcal{K}$, then π decomposes into a direct sum

$$\pi=\pi_0\oplus\pi_1 \text{ on } \mathcal{K}=\mathcal{K}_0\oplus\mathcal{K}_1,$$

where π_0 and π_1 are disjoint representations of $C^*(B_1,\dots,B_n)$ on the Hilbert spaces

$$\mathcal{K}_0 := \left\{x\in\mathcal{K}:\ \lim_{k\to\infty}\left\langle \Phi_{p,V}^k(I_{\mathcal{K}})x,x\right\rangle=0\right\}\quad \text{ and }$$
$$\mathcal{K}_1 := \left\{x\in\mathcal{K}:\ \left\langle \Phi_{p,V}^k(I_{\mathcal{K}})x,x\right\rangle=\|x\|^2\ \text{ for any } k\in\mathbb{N}\right\},$$

respectively, where $V_i:=\pi(B_i)$, $i=1,\dots,n$. Moreover, up to an isomorphism,

$$\mathcal{K}_0\simeq \mathcal{N}_{\mathcal{P}}\otimes\mathcal{G},\quad \pi_0(X)=X\otimes I_{\mathcal{G}}\quad \text{ for }\ X\in C^*(B_1,\dots,B_n),$$

where $\mathcal{G}$ is a Hilbert space with

$$\dim\mathcal{G}=\dim\left\{\operatorname{range}\left[I_{\mathcal{K}}-\Phi_{p,V}(I_{\mathcal{K}})\right]\right\},$$

and π_1 is a $*$-representation which annihilates the compact operators and $\Phi_{p,\pi_1(B)}(I_{\mathcal{K}_1})=I_{\mathcal{K}_1}$, where $\pi_1(B):=(\pi_1(B_1),\dots,\pi_1(B_1))$. Moreover, the decomposition is essentially unique.

Note that the decomposition above coincides with the one provided by Corollary 4.6 when $(T_1,\dots,T_n)=(V_1,\dots,V_n)$.

We need a few more definitions. Let $\mathcal{P}$ be a set of noncommutative polynomials. We say that an n-tuple of operators $T:=(T_1,\dots,T_n)\in\mathcal{V}_{f,\mathcal{P}}^1(\mathcal{H})$ is of class $C_{\cdot 0}$ (or pure) if

$$\lim_{k\to\infty}\left\langle\Phi_{f,T}^k(I)h,h\right\rangle=0\quad\text{for any}\quad h\in\mathcal{H},$$

and of class $C_{\cdot 1}$ if

$$\lim_{k\to\infty}\left\langle\Phi_{f,T}^k(I)h,h\right\rangle\neq 0\quad\text{for any}\quad h\in\mathcal{H},\ h\neq 0.$$

We say that $T:=(T_1,\dots,T_n)\in\mathcal{V}_{f,\mathcal{P}}^1(\mathcal{H})$ has a triangulation of type $C_{\cdot 0}-C_{\cdot 1}$ if there is an orthogonal decomposition $\mathcal{H}=\mathcal{H}_0\oplus\mathcal{H}_1$ with respect to which

$$T_i=\begin{pmatrix} C_i & 0\\ * & D_i\end{pmatrix},\qquad i=1,\dots,n,$$

and the entries have the following properties:

(i) $T_i^*\mathcal{H}_0\subseteq\mathcal{H}_0$ for any $i=1,\dots,n$;
(ii) $(C_1,\dots,C_n)\in\mathcal{V}_{f,\mathcal{P}}^1(\mathcal{H}_0)$ is of class $C_{\cdot 0}$;
(iii) $(D_1,\dots,D_n)\in\mathcal{V}_{f,\mathcal{P}}^1(\mathcal{H}_1)$ is of class $C_{\cdot 1}$.

The type of the entry denoted by $*$ is not specified.

Theorem 4.7. *Every n-tuple $T:=(T_1,\dots,T_n)\in\mathcal{V}_{f,\mathcal{P}}^1(\mathcal{H})$ has a triangulation of type*

$$\begin{pmatrix} C_{\cdot 0} & 0\\ * & C_{\cdot 1}\end{pmatrix}$$

Moreover, this triangulation is uniquely determined.

Proof. First, note that due to Theorem 4.4, the subspace

$$\mathcal{H}_0:=\left\{h\in\mathcal{H}:\ \lim_{k\to\infty}\left\langle\Phi_{f,T}^k(I_{\mathcal{H}})h,h\right\rangle=0\right\}$$

is invariant under each operator T_i^*, $i=1,\dots,n$. The decomposition $\mathcal{H}=\mathcal{H}_0\oplus\mathcal{H}_1$, where $\mathcal{H}_1:=\mathcal{H}\ominus\mathcal{H}_0$, yields the triangulations

$$T_i^*=\begin{pmatrix} C_i^* & *\\ 0 & D_i^*\end{pmatrix},\qquad i=1,\dots,n,$$

where $C_i^*:=T_i^*|_{\mathcal{H}_0}$ and $D_i^*:=P_{\mathcal{H}_1}T_i^*|_{\mathcal{H}_1}$ for each $i=1,\dots,n$. Since $T_i^*(\mathcal{H}_0)\subseteq\mathcal{H}_0$, $i=1,\dots,n$, we have

$$\Phi_{f,C}(I_{\mathcal{H}_0})=P_{\mathcal{H}_0}\Phi_{f,T}(I_{\mathcal{H}})|_{\mathcal{H}_0}\leq I_{\mathcal{H}_0}$$

and $p(C_1,\dots,C_n)=P_{\mathcal{H}_0}p(T_1,\dots,T_n)|_{\mathcal{H}_0}=0$ for any $p\in\mathcal{P}$. Therefore, $(C_1,\dots,C_n)\in\mathcal{V}_{f,\mathcal{P}}^1(\mathcal{H}_0)$. On the other hand, we have

$$\lim_{k\to\infty}\left\langle\Phi_{f,C}^k(I_{\mathcal{H}_0})h,h\right\rangle=\lim_{k\to\infty}\left\langle\Phi_{f,T}^k(I_{\mathcal{H}})h,h\right\rangle=0,\quad h\in\mathcal{H}_0,$$

which shows that the n-tuple $C := (C_1, \dots, C_n)$ is of class $C_{\cdot 0}$. Now, due to the fact that $T_i(\mathcal{H}_1) \subseteq \mathcal{H}_1$, $i = 1, \dots, n$, and $\Phi_{f,T}$ is a positive map, we have

$$\Phi_{f,D}(I_{\mathcal{H}_1}) = P_{\mathcal{H}_1}\Phi_{f,T}(P_{\mathcal{H}_1})|_{\mathcal{H}_1} \le P_{\mathcal{H}_1}\Phi_{f,T}(I_{\mathcal{H}})|_{\mathcal{H}_1} \le I_{\mathcal{H}_1}$$

and $p(D_1, \dots, D_n) = p(T_1, \dots, T_n)|_{\mathcal{H}_1} = 0$ for any $p \in \mathcal{P}$. Therefore, $(D_1, \dots, D_n) \in \mathcal{V}^1_{f,\mathcal{P}}(\mathcal{H}_1)$. We need to show that

$$\lim_{k\to\infty} \left\langle \Phi^k_{f,D}(I_{\mathcal{H}_1})h, h\right\rangle \ne 0 \quad \text{ for all } \quad h \in \mathcal{H}_1, h \ne 0.$$

Taking into account that $\Phi^k_{f,T}(I)P_{\mathcal{H}_0} \to 0$ strongly, as $k \to \infty$, $\|\Phi^k_{f,T}(I)P_{\mathcal{H}_0}\| \le 1$ for $k \in \mathbb{N}$, and $\Phi_{f,T}$ is WOT -continuous on bounded sets, we deduce that

$$\lim_{k\to\infty} \left\langle \Phi^q_{f,T}\left(\Phi^k_{f,T}(I)P_{\mathcal{H}_0}\right)h, h'\right\rangle = 0, \qquad h, h' \in \mathcal{H},$$

for each $q \ge 1$. Hence, using the fact that $Q_{f,T} := \text{SOT-}\lim_{k\to\infty} \Phi^k_{f,T}(I)$, we have

$$\begin{aligned}
\langle Q_{f,T}h, h'\rangle &= \lim_{k\to\infty} \left\langle \Phi^q_{f,T}\left(\Phi^k_{f,T}(I)\right))h, h'\right\rangle \\
&= \lim_{k\to\infty} \left\langle \Phi^q_{f,T}\left(\Phi^k_{f,T}(I)P_{\mathcal{H}_0}\right)h, h'\right\rangle + \lim_{k\to\infty} \left\langle \Phi^q_{f,T}\left(\Phi^k_{f,T}(I)P_{\mathcal{H}_1}\right)h, h'\right\rangle \\
&= \lim_{k\to\infty} \left\langle \Phi^q_{f,T}\left(\Phi^k_{f,T}(I)P_{\mathcal{H}_1}\right)h, h'\right\rangle = \left\langle \Phi^q_{f,T}\left(Q_{f,T}P_{\mathcal{H}_1}\right)h, h'\right\rangle
\end{aligned} \tag{4.1}$$

for any $h, h' \in \mathcal{H}$. Now, we need to prove that

$$\| \, Q_{f,T}h\| \le \left\langle \Phi^q_{f,T}(P_{\mathcal{H}_1})h, h\right\rangle^{1/2}, \qquad h \in \mathcal{H}.$$

First, recall that $\|\Phi^k_{f,T}(Q_{f,T}P_{\mathcal{H}_1})\| \le 1$, $k \in \mathbb{N}$, and $\Phi_{f,T}$ is WOT -continuous on bounded sets. Consequently, given $h, h' \in \mathcal{H}$, the expression $\left\langle \Phi^k_{f,T}(Q_{f,T}P_{\mathcal{H}_1})h, h'\right\rangle$ can be approximated by sums of type

$$\Sigma := \sum_{|\alpha_q|\le N_q} \cdots \sum_{|\alpha_1|\le N_1} \left\langle a_{\alpha_q}\cdots a_{\alpha_1}T_{\alpha_q}\cdots T_{\alpha_1}(Q_{f,T}P_{\mathcal{H}_1})T^*_{\alpha_1}\cdots T^*_{\alpha_q}h, h'\right\rangle,$$

where $N_1, \dots, N_q \in \mathbb{N}$. Since $\|Q_{f,T}\| \le 1$, $a_\alpha \ge 0$, we obtain

$$\begin{aligned}
&\left|\left\langle a_{\alpha_q}\cdots a_{\alpha_1}T_{\alpha_q}\cdots T_{\alpha_1}(Q_{f,T}P_{\mathcal{H}_1})T^*_{\alpha_1}\cdots T^*_{\alpha_q}h, h'\right\rangle\right| \\
&\qquad \le a_{\alpha_q}\cdots a_{\alpha_1}\left\|(Q_{f,T}P_{\mathcal{H}_1})T^*_{\alpha_1}\cdots T^*_{\alpha_q}h\right\|\left\|T^*_{\alpha_1}\cdots T^*_{\alpha_q}h'\right\| \\
&\qquad \le a_{\alpha_q}\cdots a_{\alpha_1}\left\|P_{\mathcal{H}_1}T^*_{\alpha_1}\cdots T^*_{\alpha_q}h\right\|\left\|T^*_{\alpha_1}\cdots T^*_{\alpha_q}h'\right\|.
\end{aligned}$$

Applying Cauchy's inequality, we get

$$\begin{aligned}
|\Sigma| \le &\left(\sum_{|\alpha_q|\le N_q} \cdots \sum_{|\alpha_1|\le N_1} \left\langle a_{\alpha_q}\cdots a_{\alpha_1}T_{\alpha_q}\cdots T_{\alpha_1}(P_{\mathcal{H}_1})T^*_{\alpha_1}\cdots T^*_{\alpha_q}h, h\right\rangle\right)^{1/2} \\
&\times \left(\sum_{|\alpha_q|\le N_q} \cdots \sum_{|\alpha_1|\le N_1} \left\langle a_{\alpha_q}\cdots a_{\alpha_1}T_{\alpha_q}\cdots T_{\alpha_1}T^*_{\alpha_1}\cdots T^*_{\alpha_q}h', h'\right\rangle\right)^{1/2}.
\end{aligned}$$

Taking the limits as $N_1 \to \infty, \dots, N_q \to \infty$, we obtain

$$\begin{aligned}
\left|\left\langle \Phi^q_{f,T}(Q_{f,T}P_{\mathcal{H}_1})h, h'\right\rangle\right| &\le \left\langle \Phi^q_{f,T}(P_{\mathcal{H}_1})h, h\right\rangle^{1/2}\left\langle \Phi^q_{f,T}(I)h', h'\right\rangle^{1/2} \\
&\le \left\langle \Phi^q_{f,T}(P_{\mathcal{H}_1})h, h\right\rangle^{1/2}\|h'\|
\end{aligned}$$

for any $h, h' \in \mathcal{H}$. Hence, we deduce that

$$\left\|\Phi^q_{f,T}(Q_{f,T}P_{\mathcal{H}_1})h\right\| \le \left\langle \Phi^q_{f,T}(P_{\mathcal{H}_1})h, h\right\rangle^{1/2}, \qquad h \in \mathcal{H}, q \in \mathbb{N}.$$

Combining this inequality with relation (4.1), we obtain

$$\|Q_{f,T}h\| \leq \left\langle \Phi^q_{f,T}(P_{\mathcal{H}_1})h, h\right\rangle^{1/2} = \left\langle \Phi^q_{f,D}(I_{\mathcal{H}_1})h, h\right\rangle^{1/2}, \qquad h \in \mathcal{H}, q \in \mathbb{N}.$$

Let $h \in \mathcal{H}_1$, $h \neq 0$, and assume that $\Phi^q_{f,D}(I_{\mathcal{H}_1})h \to 0$, as $q \to \infty$. The above inequality shows that $Q_{f,T}h = 0$, i.e., $h \in \mathcal{H}_0$, which is a contradiction.

Now, we prove the uniqueness. Assume that there is another decomposition $\mathcal{H} = \mathcal{M}_0 \oplus \mathcal{M}_1$ which yields the triangulations

$$T_i = \begin{pmatrix} E_i & 0 \\ * & F_i \end{pmatrix}, \qquad i = 1, \ldots, n,$$

of type $\begin{pmatrix} C_{\cdot 0} & 0 \\ * & C_{\cdot 1} \end{pmatrix}$, where ${E_i}^* := T_i^*|_{\mathcal{M}_0}$ and ${F_i}^* := P_{\mathcal{M}_1}T_i^*|_{\mathcal{M}_1}$ for each $i = 1, \ldots, n$. To prove uniqueness, it is enough to show that $\mathcal{H}_0 = \mathcal{M}_0$. Notice that if $h \in \mathcal{M}_0$, then, due to the fact that $(E_1, \ldots, E_n)$ is of class $C_{\cdot 0}$, we have

$$\lim_{k\to\infty} \left\langle \Phi^k_{f,T}h, h\right\rangle = \lim_{k\to\infty} \left\langle \Phi^k_{f,E}h, h\right\rangle = 0.$$

Hence, $h \in \mathcal{H}_0$, which proves that $\mathcal{M}_0 \subseteq \mathcal{H}_0$. Assume now that $h \in \mathcal{H}_0 \ominus \mathcal{M}_0$. Since $h \in \mathcal{M}_1$, we have

$$\lim_{k\to\infty} \left\langle \Phi^k_{f,F}h, h\right\rangle = \lim_{k\to\infty} \left\langle \Phi^k_{f,T}(P_{\mathcal{M}_1})h, h\right\rangle \leq \lim_{k\to\infty} \left\langle \Phi^k_{f,T}(I)h, h\right\rangle = 0.$$

Consequently, since $(F_1, \ldots, F_n)$ is of class $C_{\cdot 1}$, we must have $h = 0$. Hence, we deduce that $\mathcal{H}_0 \ominus \mathcal{M}_0 = \{0\}$, which shows that $\mathcal{M}_0 = \mathcal{H}_0$. This completes the proof. □

Corollary 4.8. *If $T := (T_1, \ldots, T_n) \in \mathcal{V}^1_{f,\mathcal{P}}(\mathcal{H})$ is such that $T \notin C_{\cdot 0}$ and $T \notin C_{\cdot 1}$, then there is a non-trivial joint invariant subspace under the operators $T_1, \ldots, T_n$.*

We say that $T := (T_1, \ldots, T_n) \in \mathcal{V}^1_{f,\mathcal{P}}(\mathcal{H})$ is of class C_c if

$$\left\langle \Phi^k_{f,T}(I)h, h\right\rangle = \|h\|^2 \quad \text{for any} \quad h \in \mathcal{H}, k \in \mathbb{N},$$

and of class C_{cnc} if for each $h \in \mathcal{H}$, $h \neq 0$, there exists $k \in \mathbb{N}$ such that

$$\left\langle \Phi^k_{f,T}(I)h, h\right\rangle \neq \|h\|^2.$$

We say that $T := (T_1, \ldots, T_n) \in \mathcal{V}^1_{f,\mathcal{P}}(\mathcal{H})$ has a triangulation of type $C_c - C_{cnc}$ if there is an orthogonal decomposition $\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_{cnc}$ with respect to which

$$T_i = \begin{pmatrix} C_i & 0 \\ * & D_i \end{pmatrix}, \qquad i = 1, \ldots, n,$$

and the entries have the following properties:

(i) $T_i^*\mathcal{H}_c \subseteq \mathcal{H}_c$ for any $i = 1, \ldots, n$;
(ii) $(C_1, \ldots, C_n) \in \mathcal{V}^1_{f,\mathcal{P}}(\mathcal{H}_c)$ is of class C_u;
(iii) $(D_1, \ldots, D_n) \in \mathcal{V}^1_{f,\mathcal{P}}(\mathcal{H}_{cnc})$ is of class C_{cnc}.

Theorem 4.9. *Let $\mathcal{P}$ be a set of noncommutative polynomials. Every n-tuple of operators $T := (T_1, \ldots, T_n) \in \mathcal{V}^1_{f,\mathcal{P}}(\mathcal{H})$ has a triangulation of type*

$$\begin{pmatrix} C_c & 0 \\ * & C_{cnc} \end{pmatrix}.$$

Moreover, this triangulation is uniquely determined.

Proof. Consider the subspace $\mathcal{H}_c \subseteq \mathcal{H}$ defined by

$$\mathcal{H}_c := \left\{h \in \mathcal{H} : \left\langle \Phi^k_{f,T}h, h\right\rangle = \|h\|^2 \text{ for any } k \in \mathbb{N}\right\}.$$

The fact that $\mathcal{H}_c$ is invariant under each operator $T_1^*, \ldots, T_n^*$ is due to Lemma 4.3 and Theorem 4.4. Consequently, we have the following triangulation with respect to the decomposition $\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_{cnc}$,

$$T_i = \begin{pmatrix} C_i & 0 \\ * & D_i \end{pmatrix}, \qquad i = 1, \ldots, n,$$

where $C_i^* := T_i^*|_{\mathcal{H}_c}$ and $D_i^* := P_{\mathcal{H}_{cnc}} T_i^*|_{\mathcal{H}_{cnc}}$ for each $i = 1, \dots, n$. As in the proof of Theorem 4.7, taking into account that $T_i^*(\mathcal{H}_c) \subseteq \mathcal{H}_c$ and $T_i(\mathcal{H}_{cnc}) \subseteq \mathcal{H}_{cnc}$ for each $i = 1, \dots, n$, we can show that $(C_1, \dots, C_n) \in \mathcal{V}^1_{f,\mathcal{P}}(\mathcal{H}_c)$ and $(D_1, \dots, D_n) \in \mathcal{V}^1_{f,\mathcal{P}}(\mathcal{H}_{cnc})$. Since

$$\left\langle \Phi^k_{f,C}(I_{\mathcal{H}_c})h, h\right\rangle = \left\langle \Phi^k_{f,T}(I_{\mathcal{H}})h, h\right\rangle = \|h\|^2, \qquad h \in \mathcal{H}_c, k \in \mathbb{N},$$

the n-tuple $(C_1, \dots, C_n)$ is of class C_u. Now, we need to show that $(D_1, \dots, D_n)$ is of class C_{cnc}. To this end, let $h \in \mathcal{H}_{cnc}$, $h \neq 0$, and assume that $\left\langle \Phi^k_{f,D}(I_{\mathcal{H}})h, h\right\rangle = \|h\|^2$ for all $k \in \mathbb{N}$. Then, we have

$$\begin{aligned} \|h\|^2 &= \left\langle \Phi^k_{f,D}(I_{\mathcal{H}})h, h\right\rangle = \left\langle \Phi^k_{f,T}(P_{\mathcal{H}_{cnc}})h, h\right\rangle \\ &\leq \left\langle \Phi^k_{f,T}(I_{\mathcal{H}})h, h\right\rangle \leq \|h\|^2. \end{aligned}$$

Consequently, $\left\langle \Phi^k_{f,T}(I_{\mathcal{H}})h, h\right\rangle = \|h\|^2$ for all $k \in \mathbb{N}$. Since $h \in \mathcal{H}_{cnc}$, we must have $h = 0$. This proves that $(D_1, \dots, D_n)$ is of class C_{cnc}. The uniqueness of the triangulation can be proved as in Theorem 4.7. We leave it to the reader. The proof is complete. ☐

Corollary 4.10. *If $T := (T_1, \dots, T_n) \in \mathcal{V}^1_{f,\mathcal{P}}(\mathcal{H})$ is such that $\Phi_{f,T}(I) \neq I$ and there is a non-zero vector $h \in \mathcal{H}$ such that $\left\langle \Phi^k_{f,T} h, h\right\rangle = \|h\|^2$ for any $k \in \mathbb{N}$, then there is a non-trivial invariant subspace under the operators $T_1, \dots, T_n$.*

Note that $C_c \subseteq C_{\cdot 1}$. Combining Theorem 4.7 with Theorem 4.9, we obtain another triangulation for n-tuples of operators in $\mathcal{V}^1_{f,\mathcal{P}}(\mathcal{H})$, that is,

$$\begin{pmatrix} C_{\cdot 0} & 0 & 0 \\ * & C_c & 0 \\ * & * & C_{cnc} \cap C_{\cdot 1} \end{pmatrix}.$$

According to Corollary 3.4, we have an analogue of Foiaş [9] and de Branges–Rovnyak [6] model theorem, for n-tuples of operators $(T_1, \dots, T_n) \in \mathcal{V}^m_{f,\mathcal{P}}(\mathcal{H})$ of class $C_{\cdot 0}$. When $(A_1, \dots, A_n)$ is of class $C_{\cdot 1}$, i.e.,

$$\lim_{k\to\infty} \left\langle \Phi^k_{f,A}(I)h, h\right\rangle \neq 0 \quad \text{for any} \quad h \in \mathcal{H},\ h \neq 0,$$

we can prove the following result.

Theorem 4.11. *Let $p := \sum_{1 \leq |\alpha| \leq N} a_\alpha X_\alpha$ be a positive regular polynomial and let $\mathcal{P}$ be a set of noncommutative polynomials. If $A := (A_1, \dots, A_n) \in B(\mathcal{H})^n$ is an n-tuple of operators of class $C_{\cdot 1}$ such that $\Phi_{p,A}$ is power bounded and $q(A_1, \dots, A_n) = 0$ for all $q \in \mathcal{P}$, then there exists $(T_1, \dots, T_n) \in \mathcal{V}^m_{p,\mathcal{P}}(\mathcal{H})$ of class C_c such that*

$$A_i Y = Y T_i, \qquad i = 1, \dots, n,$$

for some one-to-one operator $Y \in B(\mathcal{H})$ with range dense in $\mathcal{H}$. If, in addition, $\mathcal{H}$ is finite dimensional, then Y is an invertible operator.

Proof. Since $\Phi_{p,A}$ is power bounded, there is $M > 0$ such that $\|\Phi^k_{p,A}\| \leq M$ for all $k \in \mathbb{N}$. Note that for each $h \in \mathcal{H}$ with $h \neq 0$, we have

$$\gamma_h := \inf_{k\in\mathbb{N}} \left\langle \Phi^k_{p,A}(I)h, h\right\rangle > 0.$$

Indeed, if we assume that $\gamma_h = 0$, then for any $\epsilon > 0$ there is $k_0 \in \mathbb{N}$ such that $\left\langle \Phi^{k_0}_{p,A}(I)h, h\right\rangle \leq \frac{\epsilon}{M}$. Since $\Phi_{p,A}$ is a positive map, we have

$$\left\langle \Phi^{q+k_0}_{f,A}(I)h, h\right\rangle \leq \|\Phi^q_{p,A}\| \left\langle \Phi^{k_0}_{p,A}(I)h, h\right\rangle \leq \epsilon \qquad \text{for any } q \in \mathbb{N}.$$

Consequently, $\lim_{k\to\infty} \left\langle \Phi^k_{p,A}(I)h, h\right\rangle = 0$, which is a contradiction with the hypothesis. Now, define

$$[h, h'] := \operatorname*{LIM}_{k\to\infty} \left\langle \Phi^k_{p,A}(I)h, h'\right\rangle, \qquad h, h' \in \mathcal{H},$$

where LIM is a Banach limit. Due to the properties of a Banach limit, we have

$$0 < \gamma_h \leq [h, h] \leq M\|h\|^2, \qquad h \in \mathcal{H}, h \neq 0,$$

and

$$[h,h] = \operatorname*{LIM}_{k\to\infty} \left\langle \Phi_{p,A}^{k+1}(I)h, h \right\rangle = \operatorname*{LIM}_{k\to\infty} \sum_{1\leq|\alpha|\leq N} a_\alpha \left\langle \Phi_{p,A}^{k}(I)A_\alpha^* h, A_\alpha^* h \right\rangle$$
$$= \sum_{1\leq|\alpha|\leq N} a_\alpha [A_\alpha^* h, A_\alpha^* h]$$

for any $h \in \mathcal{H}$. Using standard theory of bounded Hermitian bilinear maps, we find a self-adjoint bounded operator $S \in B(\mathcal{H})$ such that $[h,h'] = \langle Sh, h'\rangle$ for all $h, h' \in \mathcal{H}$. Therefore, we have

$$0 < \gamma_h \leq \langle Sh, h\rangle \leq M\|h\|^2, \qquad h \in \mathcal{H}, h \neq 0,$$

which shows that S is a one-to-one positive operator with range dense in $\mathcal{H}$. Taking into account the relations obtained above, we deduce that

$$\begin{aligned} \langle Sh, h\rangle = [h,h] &= \sum_{1\leq|\alpha|\leq N} a_\alpha [A_\alpha^* h, A_\alpha^* h] \\ &= \sum_{1\leq|\alpha|\leq N} a_\alpha \langle SA_\alpha^* h, A_\alpha^* h\rangle = \langle \Phi_{p,A}(S)h, h\rangle \end{aligned} \tag{4.2}$$

for any $h \in \mathcal{H}$. Hence, $\Phi_{p,A}(S) = S$ and $a_{g_i}\|S^{1/2}A_i^* h\|^2 \leq \|S^{1/2}h\|^2$, $h \in \mathcal{H}$, for each $i = 1, \ldots, n$. Since p is a positive regular polynomial, we have $a_{g_i} > 0$ and, consequently, it makes sense to define $Z_i : S^{1/2}(\mathcal{H}) \to \mathcal{H}$ by setting

$$Z_i(S^{1/2}h) := S^{1/2}A_i^* h, \qquad h \in \mathcal{H}.$$

Since $\|Z_i(S^{1/2}h)\| \leq \frac{1}{a_{g_i}}\|S^{1/2}h\|$, $h \in \mathcal{H}$, and $S^{1/2}$ has range dense in $\mathcal{H}$, Z_i has a unique bounded linear extension to $\mathcal{H}$, which we also denote by Z_i. Therefore, $\|Z_i x\| \leq \frac{1}{a_{g_i}}\|x\|$, $x \in \mathcal{H}$. Due to relation (4.2), we have

$$\sum_{1\leq|\alpha|\leq N} a_\alpha \left\langle Z_{\tilde\alpha}^* Z_{\tilde\alpha} S^{1/2}h, S^{1/2}h \right\rangle = \|S^{1/2}h\|^2, \quad h \in \mathcal{H}.$$

Since $S^{1/2}$ has range dense in $\mathcal{H}$ and Z_i are bounded operators on $\mathcal{H}$, we deduce that $\Phi_{p,Z^*}(I) = I$. Setting now, $T_i := Z_i^*$, $i = 1, \ldots, n$, and $Y := S^{1/2}$, we get $\Phi_{p,T}(I) = I$ and $A_i Y = Y T_i$, $i = 1, \ldots, n$. Note also that $Yq(T_1, \ldots, T_n) = q(A_1, \ldots, A_n)Y = 0$ for all $q \in \mathcal{P}$. Since Y is one-to-one, we deduce that $q(T_1, \ldots, T_n) = 0$. Therefore, $(T_1, \ldots, T_n) \in \mathcal{V}_{p,\mathcal{P}}^m(\mathcal{H})$ is of class C_c. The proof is complete. ☐

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